Coherent sheaves

Terminology. Let $\Bbbk$ be an algebraically closed field. An algebraic manifold over $\Bbbk$ is a topological space X such that every point $x \in X$ has an open neighborhood $U \ni x$ equipped with a homeomorphism $\varphi_U : U \xrightarrow{\sim} X_U$, where X_U is an affine algebraic variety considered with the Zariski topology[1], and every pair of such affine charts U, W is compatible in the sense that the homeomorphism $\varphi_W \circ \varphi_U^{-1}$ between the open sets $\varphi_U(U \cap W) \subset X_U$, $\varphi_W(U \cap W) \subset X_W$ is given in coordinates by rational functions well defined within these open sets. We write $\mathcal{O}_X$ for the sheaf of local rational functions on X with values in $\Bbbk$. A sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ on X is called to be *quasicoherent* if there exists an open affine covering $X = \bigcup U_i$ such that $\mathcal{M}(W) = \mathcal{M}(U_i) \underset{\mathcal{O}_X(U_i)}{\otimes} \mathcal{O}_X(W)$ for every open $W \subset U_i$ and all i. We write $\mathcal{O}_{\mathbb{P}_n}(d)$ for the sheaf on $\mathbb{P}_n$ whose sections over an open $U \subset \mathbb{P}_n$ are rational functions φ in the homogeneous coordinates x on $\mathbb{P}_n$ representable, for every point $p \in U$, as $\varphi = f_p / g_p$, where f_p, g_p are homogeneous *polynomials* in x with $\deg f_p - \deg g_p = d$ and $g_p(p) \neq 0$.

SHA7⋄1. Given a commutative ring K and $f_1, f_2, \dots, f_m \in K$, the *Koszul complex* $K_{f_1 f_2 \dots f_m} \overset{\text{def}}{=} \bigotimes_{i=1}^m K_{f_i}$ is the tensor product of complexes K_{f_i} consisting of just two terms situated in degrees 0, 1 and looking as $K \to K$, $x \mapsto f_i x$. Write $\Lambda = \bigoplus_{i=0}^m \Lambda^i$ for the exterior algebra of the free K-module K^m of rank m with the standard basis $\xi_1, \xi_2, \dots, \xi_m$. Show that **a)** the Koszul complex $K_{f_1 f_2 \dots f_m}$ is isomorphic to the complex $0 \to \Lambda^0 \xrightarrow{\xi} \Lambda^1 \xrightarrow{\xi} \cdots \xrightarrow{\xi} \Lambda^{m-1} \xrightarrow{\xi} \Lambda^m \xrightarrow{\xi} 0$ whose differential is the right multiplication by the element $\xi = \sum f_i \xi_i \in \Lambda^1$ **b)** if f_i does not divide zero in the quotient ring $K / (f_{i+1}, \dots f_m)$ for all i, then the only non-zero cohomology of the Koszul complex is $H^m(K_{f_1 f_2 \dots f_m}) \simeq K / (f_1, f_2, \dots, f_m)$.

SHA7⋄2. Let an affine algebraic variety X with the coordinate algebra $A = \Bbbk[X]$ be covered by principal open sets $\mathcal{D}(f_i) = \{p \in X \mid f_i(p) \neq 0\}$ for some $f_1, f_2, \dots, f_m \in A$. Show that **a)** a sequence of A-module homomorphisms is exact iff its localization in[2] f_i is exact for every i **b)** for every A-module M and any $n_1, n_2, \dots, n_m \in \mathbb{N}$, the Koszul complex $M_{f_1^{n_1} f_2^{n_2} \dots f_m^{n_m}} = M \underset{A}{\otimes} K_{f_1^{n_1} f_2^{n_2} \dots f_m^{n_m}}$ is exact **c)** the Čech complex $0 \to M \to \prod_i M_{(f_i)} \to \prod_{i<j} M_{(f_i f_j)} \to \prod_{i<j<k} M_{(f_i f_j f_k)} \to \cdots$, where $M_{(h)} \overset{\text{def}}{=} M \underset{A}{\otimes} A[h^{-1}]$ and the differential ds of a family $s = \left(s_{i_0 i_1 \dots i_p}\right) \in \prod_{i_0 < \dots < i_p} M_{(f_{i_0} f_{i_1} \cdots f_{i_p})}$ has the components

$$(ds)_{i_0 i_1 \dots i_{p+1}} = \sum_{\nu=1}^{p} (-1)^\nu s_{i_0 \dots \hat{i}_p \dots i_{p+1}} \in M_{(f_{i_0} f_{i_1} \cdots f_{i_{p+1}})},$$

is represented as a filtered colimit of exact Koszul complexes, and therefore, is exact **d)** every open affine covering of an algebraic variety is acyclic for every quasicoherent sheaf.

SHA7⋄3. Use a covering of $X = \mathbb{A}^{n+1} \smallsetminus 0$ by the charts $U_i = \{(x_0, x_1, \dots, x_n) \mid x_i \neq 0\}$ to find a basis for the vector space $\bigoplus_p H^p(X, \mathcal{O}_X)$ over $\Bbbk$.

SHA7⋄4. Verify that every sheaf $\mathcal{O}_{\mathbb{P}_n}(d)$ on $\mathbb{P}_n = \mathbb{P}(V)$ is a locally free $\mathcal{O}_{\mathbb{P}_n}$-module of rank 1 and construct the *Euler exact sequence* $0 \to \mathcal{O}_{\mathbb{P}_n} \to V \otimes_{\Bbbk} \mathcal{O}_{\mathbb{P}_n}(1) \to \mathcal{T}_{\mathbb{P}_n} \to 0$, where V is the constant sheaf of vector spaces, $\mathcal{T}_{\mathbb{P}_n}$ is the tangent sheaf (of local vector fields with rational coefficients[3]).

SHA7⋄5. On the projective space $\mathbb{P}_n$, compute all cohomologies of the following coherent sheaves: **a)** $\mathcal{O}_{\mathbb{P}_n}(d)$ **b)** $\Lambda^k \mathcal{T}_{\mathbb{P}_n}$ **c)** $\Lambda^k \Omega_{\mathbb{P}_n}$, where $\Omega_{\mathbb{P}_n} = \mathcal{T}^*_{\mathbb{P}_n} = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_n}}(\mathcal{T}_{\mathbb{P}_n}, \mathcal{O}_{\mathbb{P}_n})$ is the cotangent sheaf (of local differential 1-forms with rational coefficients).

SHA7⋄6. Show that: **a)** the ideal sheaf $\mathcal{J}$ of a rational normal cubic in $\mathbb{P}_3$ fits in the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_3}(-3)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}_3}(-2)^{\oplus 3} \to \mathcal{J} \to 0$$

b) two such curves are intersecting iff they lie in a common cubic surface.

[1] This means that $X_U \subset \Bbbk^m$ is described by a system of polynomial equations and the closed subsets of X are those described by systems of polynomial equations too. The ring $\Bbbk[X_U] \overset{\text{def}}{=} \Bbbk[x_1, x_2, \dots, x_m]/I$, where the ideal I consists of all polynomials that vanish along X_U, is called the *coordinate algebra* of affine algebraic variety X_U.

[2] That is, the result of applying the exact functor $M \mapsto M \underset{A}{\otimes} A[f_i^{-1}]$.

[3] That is, $\mathcal{T}_{\mathbb{P}_n}(U)$ consists of $\Bbbk$-linear differentiations $\mathcal{O}_{\mathbb{P}_n}(U) \to \mathcal{O}_{\mathbb{P}_n}(U)$

Individual report card of ____.
(write your name and surname)

Task 7 (12.04.2018)

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