Abelian Sheaves on Paracompact Spaces.

Terminology and notations. Given a sheaf of abelian groups F on a topological space X, write $s(x)$ for a class of a section $s \in F(U)$ in the stalk F_x over a point $x \in U$. The *support* of a sheaf F (resp. of a section $s \in F(U)$) is the set $\mathrm{supp}(F) = \{x \in X \mid F_x \neq 0\}$ (resp. $\mathrm{supp}(s) = \{x \in U \mid s(x) \neq 0\}$). A sheaf F *admits a partition of unity*, if for every section $s \in F(U)$ over an open $U \subset X$ and every open covering $U = \bigcup W_\alpha$, there exist some sections $s_\alpha \in F(U)$ with $\mathrm{supp}(s_\alpha) \subset W_\alpha$ such that for every $x \in U$, $s_\alpha(x) = 0$ for all but a finite set of α's, and $\sum_\alpha s_\alpha(x) = s(x)$ in F_x. A sheaf F on X is called to be *flabby* (resp. *soft*), if the restriction map $F(X) \to F(U)$ (resp. $F(X) \to F_Z$) is surjective for all open $U \subset X$ (resp. for all closed $Z \subset X$). A sheaf F is called to be *fine* if for any two closed subsets $Z_1, Z_2 \subset X$ there is an endomorphism $F \to F$ acting identically over some open $U_1 \supset Z_1$ and equal to the zero map over some open $U_2 \supset Z_2$.

SHA5◇1. Show that **a)** in an exact triple of sheaves $0 \to F \to G \to H \to 0$ with flabby F, the sequence of groups $0 \to F(U) \to G(U) \to H(U) \to 0$ is exact for all open $U \subset X$, and G is flabby only if H is flabby **b)** the push forward of a flabby sheaf is flabby **c)** every flabby sheaf is soft.

SHA5◇2. For a sheaf F, writte G_F for its *flabby Godement envelope*, which has $G_F(U) \overset{\text{def}}{=} \prod_{x\in U} F_x$. Show that the assignment $F \mapsto G_F$ gives an exact functor from sheaves to flabby sheaves.

SHA5◇3. For a closed embedding $\iota : Z \hookrightarrow X$ and a sheaf F on Z show that **a)** the stalk $\iota_* F_x$ equals F_x for $x \in Z$, and vanishes for $x \notin Z$ **b)** the functor ι_* is exact, and $\iota^*\iota_* \simeq \mathrm{Id}$ **c)** $H^n(X, \iota_* F) \simeq H^n(Z, F)$. **d)** every morphism of sheaves $\varphi : E \to G$ on X such that $\mathrm{supp}(G) \subset Z$ is uniquely factorized through the canonical morphism $F \to \iota_*\iota^* F$.

SHA5◇4. For an open embedding $j : U \hookrightarrow X$ show that the functor j^* is exact and $j^* j_* \simeq \mathrm{Id}$.

SHA5◇5. Given some open $U_1, U_2 \subset X$ and a sheaf F on X, construct the exact sequence of groups
$\ldots \to H^p(U_1 \cup U_2, j^*_{1\cup 2}F) \to H^p(U_1, j_1^* F) \oplus H^p(U_2, j_2^* F) \to H^p(U_1 \cap U_2, j^*_{1\cap 2} F) \to H^{p+1}(U_1 \cup U_2, j^*_{1\cup 2} F) \to \ldots,$
where $j_{\ldots}$'s mean the corresponding open embeddings.

SHA5◇6. Let X be locally compact[1] and closed subsets $Z_1, Z_2 \subset X$ be compact/ For a sheaf F on X construct an exact sequence of groups $\ldots \to H^p(Z_1 \cap Z_2, \iota^*_{1\cap 2} F) \to H^p(Z_1, \iota_1^* F) \oplus H^p(Z_2, \iota_2^* F) \to H^p(Z_1 \cup Z_2, \iota^*_{1\cup 2} F) \to$ $\to H^{p+1}(Z_1 \cap Z_2, \iota^*_{1\cap 2} F) \to \ldots$, where $\iota_{\ldots}$'s mean the corresponding closed embeddings.

SHA5◇7. Let X be locally compact and paracompact[2]. Show that **a)** a sheaf F on X is soft iff for every compact closed $Z \subset X$, open $U \supset Z$, and a section $s \in F(U)$, there exists a global section $t \in F(X)$ such that $s(x) = t(x)$ in the stalks F_x over all points x from some open subset $W \subset X$ such that $Z \subset W \subset U$ **b)** a sheaf of modules over a soft sheaf of associative rings is soft **c)** a sheaf of continuous functions on X with values in $\mathbb{R}$ or $\mathbb{C}$ is soft **d)** every soft sheaf admits a partition of unity **e)** F is soft (resp. fine) iff every point has an open neighborhood $j : U \hookrightarrow X$ such that $j^* F$ is soft (resp. fine) on U **f)** F is fine iff the sheaf of rings $\mathcal{H}om(F, F)$, which has[3] $\mathcal{H}om(F, F)(U) \overset{\text{def}}{=} \mathrm{Hom}(j^* F, j^* F)$, where $j : U \hookrightarrow X$ is the open embedding, is flabby **g)** a sheaf of smooth functions on a smooth manifold X with values in $\mathbb{R}$ or $\mathbb{C}$ is fine **h)** every fine sheaf is soft **i)** for an exact triple of sheaves $0 \to F \to G \to H \to 0$ with soft F, G is soft only if H is soft, and the sequence of stalks $0 \to F_Z \to G_Z \to H_Z \to 0$ is exact for all closed $Z \subset X$.

SHA5◇8. Let a sheaf F admit a partition of unity. Show that every open covering of X is F-acyclic, and deduce from this that F is acyclic.

SHA5◇9. For a smooth real manifold X of dimension n, show that: **a)** the sheaf of differential p-forms Ω^p is acyclic for all p **b)** the complex of sheaves $0 \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \to 0$ is an acyclic resolution for the constant sheaf $\mathbb{R}^{\sim}$.

[1] That is, Hausdorf, and every point of X has an open neighborhood with the compact closure.
[2] That is, every open covering of X contains a locally finite subcovering.
[3] By the way, check that it is a sheaf.

Individual report card of ______________________________ .
(write your name and surname)

Task 5 (1.02.2018)

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