Affine algebraic – geometric dictionary

<u>Affine varieties: their ideals and coordinate algebras.</u> Let $X = V(J) \subset \mathbb{A}^n$ be an affine algebraic variety given by some ideal $J =\subset k[x_1,x_2,\dots,x_n]$ of polynomial equations. We write $I(X)$ for the ideal of all polynomials vanishing along X. If k is algebraically closed, then by Hilbert's Nullstellensatz

$$I(X) = \sqrt{J} \stackrel{\text{def}}{=} \{\, f \in k[x_1,x_2,\dots,x_n] \mid f^n \in J \text{ for some } n \in \mathbb{N}\}$$

is *the radical* of J. Clearly, $V(J_1) \subset V(J_2) \iff \sqrt{J_1} \supset \sqrt{J_2}$.

Exercice 12.1. Let $O \in \mathbb{A}_n$ be the origin; prove that $I(\{O\})$ is maximal ideal generated by x_i.

So, finitely generated commutative k-algebra $k[X] \stackrel{\text{def}}{=} k[x_1,x_2,\dots,x_n]/I(X)$ is *reduced*, i.e. does not contain nilpotent elements: $a^n = 0 \Rightarrow a = 0$. This algebra is called *coordinate* (or *structure*) algebra of the affine algebraic variety $X \subset \mathbb{A}^n$. Geometrically, $k[X]$ consists of functions $X \xrightarrow{v \mapsto f(v)} k$ obtained by restricting the polynomials $f \in k[x_1,x_2,\dots,x_n]$ onto $X \subset \mathbb{A}^n$; such the functions are called *regular* algebraic functions on X.

<u>Claim 12.2</u>. *Each reduced finitely generated algebra A over an algebraically closed field can be realized as $A = k[X]$ for some affine algebraic variety X.*

Proof. Since $A = k[x_1,x_2,\dots,x_n]/I$ is reduced, $f^n \in I \Rightarrow f \in I$ for any $f \in k[x_1,x_2,\dots,x_n]$. By Hilbert, this means that $X = V(I) \subset \mathbb{A}^n$ has $I(X) = I$ and $A = k[x_1,x_2,\dots,x_n]/I(X) = k[X]$. □

<u>Points.</u> Given a point $p \in X$, the evaluation map $\mathrm{ev}_p : k[X] \xrightarrow{f \mapsto f(p)} k$ coincides with the factorization $k[X] \xrightarrow{f \mapsto f \,(\mathrm{mod}\, \mathfrak{m}_p)} k[X]/\mathfrak{m}_p$, where $\mathfrak{m}_p \stackrel{\text{def}}{=} \{f \in k[X] \mid f(p) = 0\}$. Hence, $\mathfrak{m}_p = \ker(\mathrm{ev}_p)$ is a proper maximal ideal in $k[X]$. It is called *a maximal ideal* of p.

<u>Claim 12.3</u>. *If k is algebraically closed, then the correspondences $p \longleftrightarrow \mathrm{ev}_p \longleftrightarrow \mathfrak{m}_p$ establish bijections between the points of X, the homomorphisms $k[X] \longrightarrow k$ identical on k, and the proper maximal ideals in $k[X]$.*

Proof. The second correspondence is obviously bijective. Clearly, $p \neq q \Rightarrow \mathfrak{m}_p \neq \mathfrak{m}_q$, because there is, say, a linear form $\mathbb{A}^n \xrightarrow{\ell} k$ such that $\ell \in \mathfrak{m}_p$ but $\ell \notin \mathfrak{m}_q$. Finally, consider a proper maximal ideal $\mathfrak{m} \subset k[X]$ and write $k[X]$ as $k[x_1,x_2,\dots,x_n]/I(X)$. A full preimage $\widetilde{\mathfrak{m}} \subset k[x_1,x_2,\dots,x_n]$ of $\mathfrak{m}$ is a proper maximal ideal, because of $\widetilde{\mathfrak{m}} \supset I(X)$ and $k[x_1,x_2,\dots,x_n]/\widetilde{\mathfrak{m}} = k[X]/\mathfrak{m} = k$. Hence, $V(\widetilde{\mathfrak{m}}) \subset \mathbb{A}^n$ is nonempty and is contained in X, i.e. there is a point $p \in X$ such that $f(p) = 0$ for any $f \in \mathfrak{m}$. So, $\mathfrak{m} \subset \mathfrak{m}_p$ and $\mathfrak{m} = \mathfrak{m}_p$, because $\mathfrak{m}$ is maximal. □

<u>Algebraic varieties via spectra.</u> A set of all proper maximal ideals in a given k-algebra A is called *a maximal spectrum* of A and is denoted by $\mathit{Spec}_m(A)$. We can treat an affine algebraic variety over an algebraically closed field k pure algebraically as a maximal spectrum $\mathit{Spec}_m(A)$ of an arbitrary finitely generated reduced k-algebra A whose elements $f \in A$ are considered as the functions $\mathit{Spec}_m(A) \xrightarrow{\mathfrak{m} \mapsto f \,(\mathrm{mod}\, \mathfrak{m})} k$.

<u>Regular morphisms of algebraic varieties.</u> Any map of sets $X \xrightarrow{\varphi} Y$ produces *a pull back homomorphism* φ^* from the algebra off all k-valued functions on Y to the one on X; it sends $Y \xrightarrow{f} k$ to the composition

$$\varphi^* f \stackrel{\text{def}}{=} f \circ \varphi : X \xrightarrow{\varphi} Y \xrightarrow{f} k .$$

A map $X \xrightarrow{\varphi} Y$ between affine algebraic varieties is called *a regular morphism* of algebraic varieties, if φ^* sends the regular algebraic functions on Y to the regular algebraic functions on X, i.e. induces a well defined homomorphism of coordinate algebras $k[Y] \xrightarrow{\varphi^*} k[X]$.

<u>Claim 12.4</u>. *Let A, B be finitely generated reduced algebras over any algebraically closed field k. Then each homomorphism $B \xrightarrow{\psi} A$ such that $\psi(1) = 1$ is a pull back homomorphism: $\psi = \varphi^*$ for a unique regular map $\mathit{Spec}_m(A) \xrightarrow{\varphi} \mathit{Spec}_m(B)$; this map sends a maximal ideal $\mathfrak{m} \subset A$ to its full preimage ${\varphi^*}^{-1}(\mathfrak{m}) \subset B$.*

Proof. Let $\mathit{Spec}_m(A) \xrightarrow{\varphi} \mathit{Spec}_m(B)$ be a regular morphism, $p \in \mathit{Spec}_m(A)$ be a point, and $f \in B$ be a function on $\mathit{Spec}_m(B)$. Then $f(\varphi(p)) = 0 \iff \varphi^* f(p) = 0$, i.e. $f \in \mathfrak{m}_{\varphi(p)} \iff \varphi^*(f) \in \mathfrak{m}_p$. So, if $B \xrightarrow{\psi} A$ is a pull back of some $\mathit{Spec}_m(A) \xrightarrow{\varphi} \mathit{Spec}_m(B)$, then φ has to send $\mathfrak{m}_p \longmapsto \psi^{-1}(\mathfrak{m}_p)$ for each $p \in \mathit{Spec}_m(A)$. On the other hand, $\psi^{-1}(\mathfrak{m}) \subset B$ is proper maximal ideal for any proper maximal $\mathfrak{m} \subset A$, because of $\varphi^*(1) = 1 \Rightarrow B/\psi^{-1}(\mathfrak{m}) = \mathrm{im}\,(\psi)/(\mathfrak{m} \cap \mathrm{im}\,(\psi)) \simeq k$.

So, $\varphi\,:\, Spec_m\,(A) \xrightarrow{\mathfrak{m}_p \mapsto \psi^{-1}(\mathfrak{m}_p)} Spec_m\,(B)$ is well defined map of sets. To compute its pull back homomorphism, note that for any $b \in B$, $p \in Spec_m\,(A)$ we have

$$\varphi^* b(p) = b(\varphi(p)) = b\,\left(\bmod\, \mathfrak{m}_{\varphi(p)}\right) = b\,\left(\bmod\, \psi^{-1}(\mathfrak{m}_p)\right) = \psi(b)\,\left(\bmod\, \mathfrak{m}_p\right) = \psi b(p)\;.$$

So, $\varphi^* = \psi$ as required. □

<u>Geometric schemes.</u> Let A be an arbitrary algebra over a field k. The set of all its nilpotents $\mathfrak{n}(A) \subset A$ is an ideal, which is contained in any proper maximal ideal $\mathfrak{m} \subset A$, because $A/\mathfrak{m} = k$ has no nilpotent elements. So, a factor algebra $A_{\text{red}} \overset{\text{def}}{=} A/\mathfrak{n}$ is reduced and has the same maximal spectrum $X = Spec_m\, A = Spec_m\, A_{\text{red}}$. If k is algebraically closed, then the intersection of all maximal ideals in A_{red} is zero, because it consists of all functions vanishing everywhere on the affine algebraic variety X. Hence, the intersection of all maximal ideals in A coincides with $\mathfrak{n}(A)$.

Exercice 12.5. Show that in general situation, where A is an arbitrary commutative ring, $\mathfrak{n}(A)$ coincides with the intersection of all proper *prime*[1] ideals $\mathfrak{p} \subset A$

A pair $(A, Spec_m\, A)$, where A is an arbitrary finitely generated algebra over algebraically closed field, is called *an affine geometrical scheme*. Affine algebraic variety $X = Spec_m\, A = Spec_m\, A_{\text{red}}$ is called *a support* of this scheme. Intuitively, the scheme differs from X by allowing some «infinitezimals», that is «nilpotent functions», whose «numerical values« vanish everywhere on X. Usually, these nilpotent elements encode some «multiplicities» on X.

<u>Example 12.6</u>: an intersection of affine algebraic varieties $X, Y \subset \mathbb{A}^n$ is defined as $V(I(X) + I(Y))$, i. e. by the union of all equations for X and Y. If the intersection is non-transversal, a factor algebra

$$A = k[x_1, x_2, \dots, x_n]/(I(X) + I(Y))$$

is not reduced: say, if $X = V(x)$, $Y = V(x^2 - y) \subset \mathbb{A}_2 = Spec_m\,(k[x,y])$, then $k[x,y]/(x, y^2 - x) \simeq k[y]/(y^2)$ has quadratic nilpotent y.

Of course, geometrically, the intersection points are exhausted by an algebraic variety $Spec_m\, A_{\text{red}}$ (for $A = k[y]/(y^2)$ it consists of unique point $Spec_m\,(k)$). But a significant information will be lost under the replacement of A by A_{red}. If we want, say, to develop an intersection multiplicities technique, then we have to treat intersections as *schemes* rather than as algebraic varieties and investigate the difference between A and A_{red}.

By the definition, a *regular morphism* $(A, Spec_m\, A) \xrightarrow{(\varphi^*, \varphi)} (B, Spec_m\, B)$ of schemes is a pair that consists of an algebra homomorphism $B \xrightarrow{A}$ and a regular map $Spec_m\,(A) \xrightarrow{\varphi} Spec_m\,(B)$ of supporting algebraic varieties such that $\varphi^* f(p) = f(\varphi(p))$ for all $f \in B$, $p \in Spec_m\,(A)$. Note that now φ^* can not be uniquely recovered from φ, because the last one does not seen nilpotent infinitesimals!

<u>A direct product of affine algebraic varieties.</u> Let A, B be finitely generated k-algebras. Then the tensor product of algebras[2] $A \otimes B$ is a finitely generated k-algebra with the multiplication

$$(A \otimes B) \times (A \otimes B) \xrightarrow{(a\otimes b,\, \alpha\otimes\beta) \mapsto a\alpha \otimes b\beta} A \otimes B\;.$$

<u>Claim 12.7</u>. *A set-theoretical product of affine algebraic varieties $X = Spec_m\, A$, $Y = Spec_m\, B$ has a natural structure of an affine algebraic variety w. r. t. which the projections $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ are regular. Namely, $X \times Y = Spec_m\,(A \otimes B)$. Moreover, it has the following universal property: for any two regular maps of affine algebraic varieties: $X \xleftarrow{\varphi} Z \xrightarrow{\psi} Y$ there exists a unique regular map $Z \xrightarrow{\varphi\times\psi} X \times Y$ fitting into commutative diagram*

$$\begin{tikzcd}
& X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X & & Y \\
& Z \arrow[ul, "\varphi"] \arrow[ur, "\psi"'] \arrow[uu, "\varphi\times\psi"] &
\end{tikzcd}$$

Proof. Let us *define* $X \times Y$ as $Spec_m\,(A \otimes B)$ and the projection maps π_X, π_Y as the regular maps whose pull-back homomorphisms are the canonical algebra inclusions

$$\alpha\,:\, A \xhookrightarrow{a \mapsto a\otimes 1} A \otimes B \xleftarrow{1\otimes b \leftarrow b} B\,:\,\beta\;,$$

[1] an ideal $\mathfrak{p} \subset A$ is called *prime*, if $A/\mathfrak{p}$ has no zero divisors

[2] as a vector space over k it coincides with the tensor product of vector spaces A, B and consists of all *finite* sums $\sum a_\nu \otimes b_\nu$ with $a_\nu \in A$, $b_\nu \in B$; for example, $k[x] \otimes k[y]$ is naturally isomorphic to $k[x,y]$

Then the universal property from the claim is nothing but the universal property characterizing the tensor product of algebras: if $Z = Spec_m(C)$, then for any two algebra homomorphisms $A \xrightarrow{\varphi^*} C \xleftarrow{\psi^*} B$ there is a unique homomorphism $A \otimes B \xrightarrow{\varphi^* \otimes \psi^*} C$ such that $(\varphi^* \otimes \psi^*) \circ \alpha = \varphi^*$ and $(\varphi^* \otimes \psi^*) \circ \beta = \psi^*$.

Exercice 12.8. Deduce this universality of the tensor product of algebras w.r.t. the algebra homomorphisms from the universality of the tensor product of vector spaces w.r.t. bilinear maps (discussed in «Tensor Guide»).

In particular, on the level of points, which are nothing but the regular morphisms from $Z = Spec_m(k)$ (or equivalently, algebra surjections onto k), we have a set-theoretical bijection

$$Spec_m(A) \times Spec_m(B) \simeq \mathrm{Hom}(Spec_m(k), X) \times \mathrm{Hom}(Spec_m(k), Y) \simeq \mathrm{Hom}(Spec_m(k), X \times Y) \simeq Spec_m(A \otimes B) .$$

It remains to show that $A \otimes B$ is reduced, i.e. we have defined an affine variety but not only a scheme. Let us write $f \in A \otimes B$ as $\sum a_\nu \otimes b_\nu$ with linearly independent (over k) functions $b_\nu \in B$. If our f produces a zero function on $X \times Y$, then $f(p,q) = 0 \ \forall (p,q) \in X \times Y \Rightarrow \sum a_\nu(p) \cdot b_\nu = 0$ in $k[Y] \Rightarrow$ all $a_\nu(p) = 0 \quad \forall p \in X \quad \Rightarrow$ all $a_\nu = 0$ in $A \Rightarrow f = 0$ in $A \otimes B$. □

Zariski topology. Any affine algebraic variety $X = Spec_m A$ admits a canonical topology whose closed sets are zeros of ideals $I \subset A$, i.e. have a form $V(I) = \{x \in X \mid f(x) = 0 \quad \forall f \in I\}$.

Exercice 12.9. Check that $V(I)$ satisfy the closed set properties, namely, $\varnothing = V(1)$; $X = V(0)$; $\bigcap V(I_\nu) = V\big(\sum I_\nu\big)$, where $\sum I_\nu$ consists of all finite sums $\sum f_\nu$ with $f_\nu \in I_\nu$; $V(I) \cup V(J) = V(IJ)$, where IJ is an ideal spanned by all products ab with $a \in I$, $b \in J$.

Since any ideal $I \subset k[X]$ is finitely generated, each closed set may be presented as a finite intersection of hypersurfaces: $V(I) = V(f_1, f_2, \dots, f_m) = \bigcap V(f_\nu)$. So, any Zariski open set is a finite union of *principal* open sets $\mathscr{D}(f) \stackrel{\text{def}}{=} X \setminus V(f) = \{x \in X \mid f(x) \neq 0\}$. Zariski topology has a pure algebraic nature: its properties are far enough from the metric topology standards, because Zariski neighborhoods express rather some divisibility conditions than any distance relations.

Example 12.10: irreducible closed sets. A topological space X is called *reducible*, if $X = X_1 \cup X_2$ for some proper closed subsets $X_1, X_2 \subset X$. It's a vapid notion for the usual metric topology, where everything is reducible. In Zariski topology, the reducibility means an existence of non zero $f_1, f_2 \in k[X]$ such that f_1 vanishes along X_1 and f_2 vanishes along X_2. Since $f_1 f_2$ vanishes everywhere, $f_1 f_2 = 0$ in $k[X]$. So, X is reducible iff $k[X]$ has zero divisors. For example, a hypersurface $\{g(x) = 0\} \subset \mathbb{A}^n$ is irreducible iff g is a power of an irreducible polynomial. Irreducible algebraic sets generalize the prime numbers in arithmetics.

Claim 12.11. *Any affine algebraic variety admits a unique finite decomposition $X = \bigcup X_i$, where $X_i \subset X$ are irreducible proper closed subsets such that $X_i \not\subset X_j \ \forall\, i \neq j$ (they are called irreducible components of X).*

Proof. A decomposition is constructed step by step. If X is reducible, the first step takes $X = Z_1 \cup Z_2$, where $Z_{1,2}$ are proper closed subsets. Let $X = \bigcup Z_\nu$ after n steps. If each Z_ν is irreducible, then for each ν and any irreducible closed subset $Y \subset X$ either $Y \cap Z_\nu = \varnothing$ or $Y \subset Z_\nu$, because of $Y = \bigcup (Z_\nu \cap Y)$. So, if we take away all Z_ν contained in some other Z_μ, then we get the required decomposition and it is unique. If there are some reducible Z_ν after n steps, $(n+1)$-th step replaces each of them by an union of two proper closed subsets. If this procedure would never stop, then it produces an infinite chain of strictly embedded subsets $X \supset Y_1 \supset Y_2 \supset \dots$, i.e. an infinite chain of strictly increasing ideals $(0) \subset I_1 \subset I_2 \subset \dots$, which does not exist in the Noetherian algebra $k[X]$. □

Example 12.12: «big» open sets. Zariski topology is week and non Hausdorf. For example, $Z \subset \mathbb{A}^1$ is Zariski closed iff Z is finite. If X is irreducible, then any non empty open $U_1, U_2 \subset X$ have a nonempty intersection, because of $X = (X \setminus U_1) \cup (X \setminus U_2)$ in the contrary case.

Exercice 12.13. Prove that $f = g$ in $k[X]$, if X is irreducible and $f|_U = g|_U$ over some open non-empty $U \subset X$.
Hint. If $U = \mathscr{D}(h)$, then $X = V(h) \cup V(f - g)$.

Example 12.14: Zariski topology on $X \times Y$ is finer than the product of Zariski topologies on X and Y, because the closed $Z \subset X \times Y$ are not exhausted by the products of closed subsets on X, Y. For example, if $X = Y = \mathbb{A}^1$, then any curve, say a hyperbola $V(xy - 1)$, is closed in Zariski topology on $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, whereas the products of closed sets on $\mathbb{A}^1$ are exhausted by finite unions of isolated points and coordinate lines.

Claim 12.15. *A regular morphism $X \xrightarrow{\varphi} Y$ of algebraic varieties is continuous in Zariski topology.*

Proof. A preimage $\varphi^{-1}(Z)$ of a closed $Z = V(I) \subset Y$ consists of all $x \in X$ such that $0 = f(\varphi(x)) = {\varphi^*}^{-1} f(x)$ for all $f \in I$. So, it coincides with the zero set of an ideal ${\varphi^*}^{-1}(I) \subset k[X]$. □

Morphism Decomposition. Let $X \xrightarrow{\varphi} Y$ be a regular morphism of affine algebraic varieties. Then, k-algebra homomorphism $k[Y] \xrightarrow{\varphi^*} k[X]$ can be decomposed as $k[Y] \twoheadrightarrow \mathrm{im}(\varphi^*) \hookrightarrow k[X]$. Since $k[X]$ is reduced, $\mathrm{im}(\varphi^*) \subset k[X]$ is reduced too, i.e. there is an affine algebraic variety $Z = Spec_m \, \mathrm{im}(\varphi^*)$ such that $X \xrightarrow{\varphi} Y$ is decomposed as $X \xrightarrow{\varphi_1} Z \xrightarrow{\varphi_2} Y$, where $k[Y] \xrightarrow{\varphi_2^*} k[Z]$ is surjective, $k[Z] \xrightarrow{\varphi_1^*} k[X]$ is injective. The injectivity of φ_1^* means that there are no non zero functions $f \in k[Z]$ vanishing along $\varphi_1(X)$, i.e. $\varphi_1(X) \subset Z$ is dense. The surjectivity of φ_2^* means that φ_2 induces an isomorphism between Z and a closed subset $V(\ker \varphi_2^*) \subset Y$ given by the ideal $\ker(\varphi_2^*) = \ker(\varphi) \subset k[Y]$ (which is automatically radical). In other words, the Zariski closure $Z = \overline{\varphi(X)} \subset Y$ is an affine algebraic variety and the maps $X \longrightarrow \overline{\varphi(X)} \hookrightarrow Y$ are regular morphisms.

<u>Dominant morphisms, closed embeddings, and finite morphisms.</u> If X is irreducible and the algebra homomorphism $k[Y] \xrightarrow{\varphi^*} k[X]$ is injective, then the corresponding morphism $X \xrightarrow{\varphi} Y$ is called *dominant*. Geometrically, this means that $\overline{\varphi(X)} = Y$. If X is reducible, then φ is called *dominant* when its restriction onto *each* irreducible component of X is dominant. A morphism $X \xrightarrow{\varphi} Y$ is called *a closed embedding*, if $k[Y] \xrightarrow{\varphi^*} k[X]$ is surjective. This means that φ identifies X with $V(\ker \varphi^*) \subset Y$.

Exercice 12.16. Show that any dominant morphism of irreducible affine varieties $X \xrightarrow{\varphi} Y$ can be decomposed as

$$X \overset{\psi}{\hookrightarrow} Y \times \mathbb{A}^m \overset{\pi}{\twoheadrightarrow} Y \,, \tag{32}$$

where ψ is a closed embedding and π is the natural projection along $\mathbb{A}^m$.

Hint. Let $A = k[X]$, $B = k[Y]$. Since A is finitely generated B-algebra via the inclusion $B \overset{\varphi^*}{\hookrightarrow} A$, there is an epimorphism of B-algebras $B[x_1, x_2, \dots, x_m] \overset{\psi}{\twoheadrightarrow} A$ for some m.

Given a regular morphism $X \xrightarrow{\varphi} Y$, then the coordinate algebra $k[X]$ can be considered as an algebra over $\varphi^*(k[Y]) = k[\overline{\varphi(X)}] \subset k[X]$. A morphism φ is called *finite*, if $k[X]$ is integer over $\varphi^*(k[Y])$. Since $k[X]$ is finitely generated as algebra over $\varphi^*(k[Y])$ (even over k), finiteness of φ means that $k[X]$ is finitely generated also as $\varphi^*(k[Y])$-*module*, i.e. there are some $f_1, f_2, \dots, f_m \in k[X]$ such that any $h \in k[X]$ can be written as $h = \sum \varphi^*(g_i)\, f_i$ for some $g_i \in k[Y]$.

<u>Claim 12.17</u>. *Let $X \xrightarrow{\varphi} Y$ be a finite morphism of affine algebraic varieties. Then $\varphi(Z)$ is closed for any closed $Z \subset X$ and induced morphism $Z \xrightarrow{\varphi|_Z} \varphi(Z)$ is finite. Moreover, if X is irreducible and $Z \neq X$, then $\varphi(Z) \neq Y$.*

Proof. Let $I = I(Z) \subset k[X]$ be the ideal of $Z \subset X$. Then $Z \xrightarrow{\varphi|_Z} Y$ has $\varphi_z^* : k[Y] \xrightarrow{\varphi^*} k[X] \longrightarrow k[X]/I$. Since $k[X]$ is finitely generated as $\varphi^*(k[Y])$-module, $k[Z] = k[X]/I$ is finitely generated as a module over

$$k[\,\overline{\varphi(Z)}\,] = \varphi|_z^*(k[Y]) = \varphi^*(k[Y])/(I \cap \varphi^*(k[Y]))\,,$$

i.e. $Z \longrightarrow \overline{\varphi(Z)}$ is a finite morphism. To prove that $\varphi(Z) = \overline{\varphi(Z)}$, we can restrict ourself onto irreducible components of Z, i.e. suppose that Z is irreducible. Let $B = k[Z]$, $A = k[\,\overline{\varphi(Z)}\,] \subset B$, and $f_1, f_2, \dots, f_m$ generate B as A-module. Since $\varphi|_z$ takes a maximal ideal $\mathfrak{m}_p \subset B$ to the maximal ideal $\mathfrak{m}_p \cap A \subset A$, a point $q \in Spec_m(A)$ does not belong to $Spec_m(B)$ iff its maximal ideal $\mathfrak{m}_q \subset A$ generates non proper ideal in B, that is $\mathfrak{m}_q \cdot B = B$. In this case we can write $f_i = \sum \beta_{i\nu} f_\nu$ for some $\beta_{i\nu} \in \mathfrak{m}_q$; that is the zero homomorphism of A-modules: $B \longrightarrow 0$, which takes each f_i to zero, can be presented in terms of the generator system $\{f_\nu\}$ by the matrix $E - (\beta_{\nu i})$. Hence, the multiplication by $\det(E - (\beta_{ij}))$ annihilates B. Since there no zero divisors in $B = k[Z]$, we get $\det(E - (\beta_{ij})) = 0$. But $\det(E - (\beta_{ij})) = 1 + \beta$ where $\beta \in \mathfrak{m}_q$. So, $1 \in \mathfrak{m}_q$ and $\mathfrak{m}_q \subset A$ is non proper as well.

To prove that $\varphi(Z) \neq Y$ for $Z \subsetneq X$, let us take a non zero function $f \in k[X]$ vanishing along Z and write the integer equation of the lowest possible degree for f over $\varphi^*(k[Y])$ as

$$f^m + \varphi^*(g_1)\, f^{m-1} + \,\cdots\, + \varphi^*(g_{m-1})\, f + \varphi^*(g_m) = 0\,.$$

Computing its left side at any $z \in Z$, we get $\varphi^*(g_m)(z) = 0$, that is $g_m(\varphi(z)) = 0$. So, if $\varphi(Z) = Y$, then $g_m \equiv 0$ along Y, i.e. $\varphi^*(g_m) = 0$ in $k[X]$. Since $k[X]$ has no zero divisors, the minimal equation above is divisible by f. Contradiction. $\square$

<u>Normal algebraic varieties.</u> If Y is irreducible, then $k[Y]$ has no zero divisors and its quotient field is well defined; it is called *the field of rational functions* on Y and is denoted $k(Y)$. Irreducible Y is called *normal*, if all rational functions $f \in k(Y)$ integer over $k[Y]$ belong to $k[Y]$. For example, affine spaces $\mathbb{A}^n$ are normal (as well as any X with factorial $k[X]$).

<u>Claim 12.18</u>. *Let $X \xrightarrow{\varphi} Y$ be a surjective finite morphism. If Y is normal, then $\varphi(U)$ is open for any open $U \subset X$ (in particular each irreducible component of X is surjectively mapped onto Y).*

Proof. We can suppose that $U = \mathscr{D}(f)$ is principal. Let $p \in \mathscr{D}(f)$; we have to find $a \in k[Y]$ such that $\varphi(p) \in \mathscr{D}(a) \subset \varphi(\mathscr{D}(f))$. To this aim consider a map

$$\psi = \varphi \times f \;:\; X \xrightarrow{p \mapsto (\varphi(p), f(p))} Y \times \mathbb{A}^1\,.$$

It is regular and finite: $k[Y] \subset k[X]$ (via φ^*), $k[Y \times \mathbb{A}^1] = k[Y][t] \xrightarrow{\psi^*} k[X]$ is the evaluation $t \longmapsto f$, and $k[X]$ is finitely generated even as $k[Y]$-module. Now let us extend $k[X]$ to the $k(Y)$-algebra $B = k(Y) \underset{k[Y]}{\otimes} k[X]$, which consists of all fractions b/a, $b \in k[X]$, $0 \neq a \in k[Y]$ with the usual equivalence $b'/a' = b''/a'' \iff b''a' - b'a''$ divides zero in $k[X]$. Since f is integer over $k[Y]$, f is algebraic over $k(Y)$, i.e. the kernel of evaluation map $k(Y)[t] \xrightarrow{t \mapsto f} B$ is the principal ideal $(\mu_f) \subset k(Y)[t]$ spanned by the minimal polynomial $\mu_f(t) = t^m + a_1 t^{m-1} + \cdots + a_{m-1}t + a_m \in k(Y)[t]$ of f over $k(Y)$.

Exercice 12.19. Show that that μ_f has coefficients in $k[Y]$, i.e. $\mu_f \in k[Y \times \mathbb{A}^1]$.

HINT. Let $g(f) = 0$ be an *integer* equation for f over $k[Y]$. Then $g(t) = \mu_f(t) \cdot h(t)$ in $k(Y)[t]$ for some monic h. Now use the Gauss – Kronecker – Dedekind lemma (see Corollary 8.10) and normality of Y.

So, $\ker \psi^* = (\mu_f)$ and $\overline{\operatorname{im}\psi} = Spec_m\left(k[Y \times \mathbb{A}^1]/(\mu_f)\right) = V(\mu_f)$. In other words, ψ gives a finite surjection of X onto the hypersurface $V(\mu_f) \subset Y \times \mathbb{A}^1$ and φ is obtained by projecting this hypersurface onto Y. In particular, the preimages of given $y \in Y$ are surjectively mapped by f onto the roots of a polynomial

$$\mu_f(y;t) = t^m + a_1(y)\, t^{m-1} + \cdots + a_m(y) \in k[t]$$

obtained by evaluating the coefficients of μ_f at $y \in Y$. Those preimages which belong to $\mathscr{D}(f)$ correspond to non-zero roots. Since $f(p) \neq 0$, the polynomial $\mu_f(\varphi(p);t)$ has a non zero root, i.e. $\mu_f(\varphi(p);t) \neq t^m$ and there is a non zero coefficient $a_i(\varphi(p)) \neq 0$. Then $\mu_f(y;t)$ has a non zero root over each $y \in \mathscr{D}(a_i) \subset Y$, i.e. $\varphi(\mathscr{D}(f)) \supset \mathscr{D}(a_i) \ni \varphi(p)$ as required.

What about irreducible components, consider the irreducible decomposition $X = \cup X_\nu$. Then

$$U_i = X \setminus \bigcup_{\nu \neq i} X_\nu = X_i \setminus \bigcup_{\nu \neq i} (X_i \cap X_\nu)$$

is open in X and dense in X_i. Since $\varphi(U_i)$ is non empty open subset of Y, $\varphi(X_i) = \overline{\varphi(U_i)} = Y$. □