Projective hypersurfaces[1]

SPACE OF HYPERSURFACES. Projective space $\mathbb{P}(S^dV^*)$ consists of all non zero d-th degree homogeneous polynomials considered up to a scalar factor. It is called *a space of degree d hypersurfaces* in $\mathbb{P}_n = \mathbb{P}(V)$. Geometrically, each polynomial whose prime factorization is $f(x) = \prod_{i=1}^{s} p_i(x)^{m_i}$ defines a zero set

$$Z_f \stackrel{\text{def}}{=} \{x\in\mathbb{P}_n \mid f(x) = 0\} = \bigcup_{i=1}^{s} m_i \cdot Z_{p_i} \;,$$

which is an union of the *irreducible components* $Z_{p_i} = \{x \in \mathbb{P}(V) \mid p_i(x) = 0\}$ counted with integer multiplicities m_i. We will also write $Z_f = m_1Z_{p_1} + m_2Z_{p_2} + \cdots + m_sZ_{p_s}$. By Exercice 8.26, each irreducible component Z_{p_i} does not admit any further decomposition into a sum of proper subsurfaces.

Exercice 9.1. Find $\dim\mathbb{P}(S^dV^*)$.

Traditionally, 1-, 2-, and 3-dimensional projective subspaces in the space of hypersurfaces are called, respectively, *pencils*, *nets* and *webs* of hypersurfaces.

Example 9.2: pencil of plane curves $\ell = (C_1C_2) \subset \mathbb{P}(S^dV^*)$ on $\mathbb{P}_2 = \mathbb{P}(V)$ is defined by any two distinct elements $C_1, C_2 \in \ell$. A curve $\lambda C_1 + \mu C_2 \in \ell$ (whose homogeneous coordinates w.r.t. the base $\{C_1, C_2\}$, of ℓ, are $(\lambda : \mu)$) is given in $\mathbb{P}_2$ by the equation $\lambda f_1(x) + \mu f_2(x) = 0$, where $f_1(x) = 0$ and $f_2(x) = 0$ are the equations of the base curves C_1, C_2. In particular, each curve from the pencil $(C_1C_2$ pass through all the intersection points $C_1 \cap C_2$. Another remarkable property: for any $p \in \mathbb{P}_2$ any pencil of plane curves contains a curve passing through p. Indeed, curves passing through a given point form a codimension 1 hyperplane in the space of curves and this hyperplane intersects each line of curves.

This leads to another fruitful proof of the Pascal theorem. Let a hexagon $p_1p_2p_3p_4p_5p_6$ be inscribed in a non singular conic C and write $x = p_3p_4 \cap p_6p_1$, $y = p_2p_3 \cap p_5p_6$, $z = p_1p_2 \cap p_4p_5$ for the intersection points of its opposite sides. Fix some 7-th point $p_7 \in C$, which differs from $p_1, p_2, \dots, p_6$, and consider a pencil of cubic curves (Q_1Q_2) spanned by 2 completely splitted cubics: $Q_1 = (p_1p_2) \cup (p_3p_4) \cup (p_5p_6)$ and $Q_2 = (p_2p_3) \cup (p_4p_5) \cup (p_6p_1)$ formed by 'opposite triples' of sides. All cubics of this pencil pass through 9 intersection points $Q_1 \cap Q_2 = \{p_1, p_2, \dots, p_6, x, y, z\}$ and at least one of them, say Q, pass through p_7 as well. Since the conic C either intersects the cubic Q in $\leqslant 6$ points or is contained in Q, we have perforce $Q = C + \text{some line}$, where the line has to pass through x, y, z.

INTERACTION WITH LINES. Let $S \subset \mathbb{P}_n$ be a surface given by a homogeneous equation $F(x) = 0$ of degree d and $\ell = (pq) \in \mathbb{P}_n$ be a line spanned by $p, q \in V$. Write $(\lambda : \mu)$ for internal homogeneous coordinates of a point $\lambda p + \mu q \in \ell$. In these coordinates, $\ell \cap S$ is given by the equation $f(\lambda, \mu) = 0$ obtained from $F(x) = 0$ by the substitution $x = \lambda p + \mu q$. By the Newton - Taylor formula,

$$f(\lambda, \mu) = F(\lambda p + \mu q) = \sum_{i=0}^{d} \lambda^i \mu^{n-i} \binom{d}{i} \widetilde{F}(p^i, q^{n-i}) \;, \tag{27}$$

where the coefficient $\widetilde{F}(p^i, q^{n-i}) \stackrel{\text{def}}{=} \operatorname{pl}(F)(\underbrace{p, p, \dots, p}_{i}, \underbrace{q, q, \dots, q}_{d-i}) = \dfrac{(d-i)!}{d!} \dfrac{\partial^i F}{\partial p^i}(q) = \dfrac{i!}{d!} \dfrac{\partial^{d-i} F}{\partial q^{d-i}}(p)$ is bihomogeneous of degree $(i, n-i)$ in (p, q). If $f(\lambda, \mu) \equiv 0$, i.e. $\widetilde{F}(p^i, q^{n-\mu}) = 0$ for all i, then $\ell \subset S$. If $f(\lambda, \mu) \not\equiv 0$, then $f(\lambda, \mu) = \prod_i (\alpha''_i \mu - \alpha'_i \lambda)^{s_i}$ is a product of linear forms[2]. Each linear form corresponds to an intersection point $\alpha = (\alpha' : \alpha'') = \alpha' p + \alpha'' q \in \ell \cap S$. The maximal power s_i such that $f(\lambda, \mu)$ is divisible by $(\alpha''_i \mu - \alpha'_i \lambda)^{s_i}$ is called *a local intersection index* between S and ℓ at α. It is denoted by $(S, \ell)_\alpha$. So, $\deg S = \sum_{\alpha \in S \cap \ell} (S, \ell)_\alpha$ as soon as $\ell \not\subset S$. i.e. a line either lies on S or intersect S in $\deg S$ points counted with multiplicities.

TANGENT LINES AND TANGENT SPACE. A line ℓ is called *a tangent line* to S, if there is a point $p \in S \cap \ell$ with $(S, \ell)_p \geqslant 2$. We say that ℓ does *touch* S at p or that p is a *tangency point*.

Claim 9.3. *For any $p \in S$ and any $q \in \mathbb{P}_n$ the line (pq) touch S at p iff $\widetilde{F}(p^{n-1}, q) = 0$.*

Proof. If $p \in S$, that is $\widetilde{F}(p^n) = F(p) = 0$, the affine version of (27) near p takes the form:

$$F(p + tq) = t\binom{d}{1}\widetilde{F}(p^{n-1}, q) + t^2\binom{d}{2}\widetilde{F}(p^{n-2}, q^2) + \cdots$$

[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

[2] namely, $f(\lambda, \mu) = \mu^d f(t, 1)$, where $t = \lambda/\mu$ and $f(t, 1) \in k[t]$; now, $f(t, 1) = \prod(t - \alpha_i)^{m_i}$

and $(S,(pq))_p$ is the maximal power of t factored out of $F(p+tq)$. It is $\geqslant 2$ iff $\widetilde{F}(p^{n-1},q)=0$. $\square$

Corollary 9.4. *The union of all tangent lines through $p\in S$ is a projective space $T_pS \overset{\text{def}}{=} \left\{ y\in\mathbb{P}_n \;\middle|\; \sum_{i=0}^{n} y_i \frac{\partial F}{\partial x_i}(p)=0 \right\}$. It is either a hyperplane or the whole of $\mathbb{P}_n$. The last happens iff $\frac{\partial F}{\partial x_i}(p)=0\ \forall i$.* $\square$

The space T_pS is called *a tangent space* to S at p. If $T_pS=\mathbb{P}_n$, then S is called *singular* at p and p is called *a singular point* of S. Otherwise p is called *a smooth point* of S. S is called *smooth*, if all its points are smooth.

Corollary 9.5. *Let q be either a smooth point on S or any point outside S. Then the apparent contour*[3] *of S visible from q is slashed by the hypersurface $S_q^{(d-1)} \overset{\text{def}}{=} \left\{ y\in\mathbb{P}_n \;\middle|\; \sum_{i=0}^{n} q_i \frac{\partial F}{\partial x_i}(y)=0 \right\}$; of degree $(d-1)$ (in particular, $\sum_{i=0}^{n} q_i \frac{\partial F}{\partial x_i}(y)\not\equiv 0$ as a polynomial in y).*

Proof. Indeed, (qy) touch S at y, if $0=\widetilde{F}(y^{n-1},q)=\mathrm{pl}_qF(y)=\frac{1}{d}\sum_{i=0}^{n} q_i\frac{\partial F}{\partial x_i}(y)$. If this polynomial vanishes identically in y, then taking $y=q$ we get $F(q)=0$, i.e. $q\in S$. At the same time $F(q,q,\dots,q,y)=\mathrm{pl}_q^{n-1}F(y)=\mathrm{pl}_q^{n-2}\mathrm{pl}_qF(y)\equiv 0$, because of $\widetilde{F}(y^{n-1},q)\equiv 0$. So, q is singular point of S. $\square$

Point multiplicities. A number $\mathsf{mult}_S(p) \overset{\text{def}}{=} \min_{\ell\ni p}(\ell,S)_p$ is called *a multiplicity* of p on S. A point $p\in S$ is singular iff any line through p intersects S with index $\geqslant 2$ at p. So, $p\in S$ is smooth iff p has the multiplicity 1. A point p has multiplicity $\geqslant m$ iff all possible $(m-1)$-type partial derivatives of F vanish at p.

Polar hypersurfaces. A hypersurface $S_q^{(r)} \overset{\text{def}}{=} \left\{ y\in\mathbb{P}_n \mid \widetilde{F}(q^{n-r},y^r)=0 \right\}$ is called *a r-th degree polar* of S with respect to p. If $F(q^{n-r},y^r)$ vanishes identically in y, we say that the polar is *trivial*, i.e. coincides with the whole of $\mathbb{P}_n$. Intuitively, for a smooth point $q\in S$, the polar $S_q^{(r)}$ is a degree r surface which gives the most closed approximation for S near q in a sense that the both have at q the same tangent hyperplanes (i.e. their linear polars at q coincide), the same 'tangent quadrics' (i.e. their quadratic polars at q coincide), and so on up to coincidence of $(r-1)$-th degree polars. If $q\in S$ is singular of multiplicity $m\geqslant 2$, then all the polars of degree $\leqslant (m-1)$ w.r.t. p are trivial and the m-th degree polar is non trivial but singular at p.

Example 9.6: space of singular conics. Let V be 3D vector space, $\mathbb{P}_5=\mathbb{P}(S^2V^*)$ be the space of conics on $\mathbb{P}_2=\mathbb{P}(V)$, and $S\subset\mathbb{P}_5$ be a locus of the singular conics. Let us fix some coordinates and present quadratic forms $q(x)\in S^2V^*$ as $q(x)=x\cdot A\cdot {}^tx$ with symmetric 3×3-matrices A. Since q is singular iff $\det A=0$, we see that S is an irreducible cubic hypersurface in $\mathbb{P}_5$. We would like to find its singular points and describe non singular tangent hyperplanes. By Laplace's relations, $\det A=\sum_{\nu}(-1)^{i+\nu}a_{i\nu}A_{i\nu}$, where $A_{i\nu}$ is 2×2-minor situated outside i-th row and j-th column. So, $\frac{\partial \det A}{\partial a_{ij}}=(-1)^{i+j}A_{ij}$ and a point $q\in S$ is singular iff $\operatorname{rk}A=1$.

Exercice 9.7. Show that any $m\times n$ matrix a_{ij} of rank 1 has $a_{ij}=\lambda_i\mu_j$ for some $\lambda_1,\lambda_2,\dots,\lambda_m$, $\mu_1,\mu_2,\dots,\mu_n\in k$.

Hint. A linear operator $k^n \xrightarrow{x\mapsto Ax} k^m$ has $\operatorname{rk}A=1$ iff $\dim\operatorname{im}A=1$; if $w=(\lambda_1,\lambda_2,\dots,\lambda_m)$ generates $\operatorname{im}A$, then $A(v)=\alpha(v)\,w$, where $k^n\xrightarrow{\alpha}k$ is linear form, say $\alpha=(\mu_1,\mu_2,\dots,\mu_n)\dots$

In our case $A=(a_{ij})$ is symmetric and we should have $\mu_i=\lambda_i$, i.e. $a_{ij}=\lambda_i\lambda_j$ for some $\lambda_0,\lambda_1,\lambda_2\in k$. So, $A\in S$ is singular iff $q(x)=\left(\sum\lambda_ix_i\right)^2$ is a double line. Thus, the set of singular points of S coincides with 2-dimensional Veronese's surface $\mathcal{V}\subset S$, which parameterizes double lines in $\mathbb{P}_2$.

Now, let $q(x)=x\cdot A\cdot {}^tx$ be a smooth point of S, i.e. a pair of distinct lines $\ell_1\cup\ell_2\subset\mathbb{P}_2$. Then the correlation map $V\xrightarrow{x\mapsto x\cdot A}V^*$ has 1-dimensional kernel spanned by $v=\ell_1\cap\ell_2$, that is $\operatorname{rk}(A)=2$ and the adjoint matrix $\widehat{A}=\left((-1)^{i+j}A_{ij}\right)$ is non zero. By the Laplace relations: $A\cdot\widehat{A}=\widehat{A}\cdot A=\det(A)\cdot\mathrm{Id}=0$, each row and each column of $\widehat{A}$ lies in the kernel of A, i.e. is proportional to v. Thus, $\operatorname{rk}\widehat{A}=1$ and $(-1)^{i+j}A_{ij}=\mu_i\mu_j$, where $(\mu_0:\mu_1:\mu_2)$ are homogeneous coordinates of v. So, $B=(b_{ij})\in T_qS_n \iff \sum_{ij} b_{ij}\cdot(-1)^{i+j}A_{ij}=0 \iff \sum_{ij} b_{ij}\,\mu_i\mu_j=0 \iff v\cdot B\cdot {}^tv=0$. In other words, the tangent space T_qS at $q=\ell_1\cup\ell_2\subset\mathbb{P}_2$ consists of all conics passing through the point $\ell_1\cap\ell_2\in\mathbb{P}_2$.

Exercice 9.8. Extend this result to general case $\dim V=n+1$, i.e. show that a point q on the hypersurface $S\subset\mathbb{P}(S^2V^*)$, of singular quadrics on $\mathbb{P}_n=\mathbb{P}(V)$, is non-singular iff the corresponding quadric $Q_q\subset\mathbb{P}(V)$ has just one singular point $v(q)\in\mathbb{P}(V)$ and prove that T_qS consists of all quadrics passing through $v(q)$.

[3] i.e. the set of all tangency points $p\neq q$ where S touched by the tangent lines drawn from q