Rational functions and maps

AG6◇1 (rational functions). Write $\Bbbk(X)$ for the algebra of rational functions on an affine algebraic variety X, that is, the algebra of fractions p/q, where $p, q \in \Bbbk[X]$ and q is not a zero divisor. For $f \in \Bbbk(X)$, the subset

$$\operatorname{Dom}(f) \overset{\text{def}}{=} \{x \in X \mid \exists\, p, q \in \Bbbk[X] : q(x) \neq 0 \;\&\; f = p/q\} \subset X$$

is called the *domain* of f. Show that:

a) for $x \in \operatorname{Dom}(f)$, the value $f(x) = p(x)/q(x) \in \Bbbk$ does not depend on a choice of fractional representation $f = p/q$ with $p, q \in \Bbbk[X]$ and $q(x) \neq 0$

b) $\operatorname{Dom}(f)$ is open and dense in X

c) the map $f : \operatorname{Dom}(f) \to \Bbbk$, $x \mapsto f(x)$, is continuous in Zariski topology.

AG6◇2. Find $\operatorname{Dom}(f)$ for the following rational functions:

a) $f = (1 - y)/x$ on $V(x^2 + y^2 - 1) \subset \mathbb{A}^2$

b) $f = y/x$ on $V(x^3 + x^2 - y^2) \subset \mathbb{A}^2$

c) $f = x_1/x_3$ on $X = V(x_1x_4 - x_2x_3) \subset \mathbb{A}^4$.

AG6◇3. Let $X = X_1 \cup X_2 \cup \ldots \cup X_m$ be the irreducible decomposition of an affine algebraic variety X. Write $f|_{X_i}$ for the image of a rational function f on X under the homomorphism $\Bbbk(X) \to \Bbbk(X_i)$ that extends the pullback homomorphism $\varphi_i^* : \Bbbk[X] \twoheadrightarrow \Bbbk[X_i]$ of the closed immersion $\varphi_i : X_i \hookrightarrow X$. Prove that the map

$$\Bbbk(X) \xrightarrow{\sim} \Bbbk(X_1) \times \Bbbk(X_2) \times \cdots \times \Bbbk(X_m), \quad f \mapsto \left(f|_{X_1}, f|_{X_2}, \ldots, f|_{X_m}\right),$$

is an isomorphism of $\Bbbk$-algebras.

AG6◇4. Prove that $\mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n \smallsetminus 0) = \Bbbk[\mathbb{A}^n]$ for $n \geqslant 2$.

AG6◇5*. Do there exist an affine algebraic variety $X \subset \mathbb{A}^n$ and an open subset $U \subset X$ such that the algebra $\mathcal{O}_X(U)$ is not finitely generated?

AG6◇6 (Cremona's quadratic involution). Show that the assignment $(t_0 : t_1 : t_2) \mapsto (t_0^{-1} : t_1^{-1} : t_2^{-1})$ can be extended to a rational map $\varkappa : \mathbb{P}_2 \dashrightarrow \mathbb{P}_2$ defined everywhere except three points. Find these points and describe the action of $\varkappa$ on the three lines joining these points. Describe the image of $\varkappa$.

AG6◇7 (the graph of a rational map). Let $\psi : X \dashrightarrow Y$ be a rational map defined on some open dense subset $U \subset X$. The Zariski closure of the set $\{(x, \psi(x)) \in X \times Y \mid x \in U\}$ is called the *graph* of ψ and denoted by

$$\Gamma_\psi \subset X \times Y.$$

a) Show that the graph of canonical projection $\mathbb{A}(V) \dashrightarrow \mathbb{P}(V)$ sending a nonzero vector $v \in V$ to the dimension-1 subspace $\Bbbk \cdot v \subset V$ coincides with the blowup of $\mathbb{A}(V)$ at the origin.

b) Describe the graph $\Gamma_\varkappa \subset \mathbb{P}_2 \times \mathbb{P}_2$ of the Cremona quadratic involution from prb. AG6◇6 and the fibers of two projections of this graph to $\mathbb{P}_2$'s.

AG6◇8. Prove that the variety obtained from $\mathbb{P}_2$ by blowing up two different points on $\mathbb{P}_2$ is isomorphic to the blowup of $\mathbb{P}_1 \times \mathbb{P}_1$ at one point.

Individual report card of ______________________________________. Task 6 (November 17, 2017)
(write your name and surname)

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