§2 Projective Quadrics

2.1 Quadratic forms and quadrics. We assume on default in §2 that $\operatorname{char}\Bbbk \neq 2$. Projective hypersurfaces of degree 2 are called *projective quadrics*. Given a non-zero quadratic form $q \in S^2V^*$, we write $Q \subset \mathbb{P}(V)$ for the quadric $Q = V(q)$.

2.1.1 The Gram matrix. If $\operatorname{char}\Bbbk \neq 2$, then every quadratic form $q \in S^2V^*$ on $V = \Bbbk^{n+1}$ can be written as $q(x) = \sum_{i,j} a_{ij}\, x_i x_j = xAx^t$, where $x = (x_0, x_1, \dots, x_n)$ is the coordinate row, x^t is the transposed column of coordinates, and $A = \left(a_{ij}\right) \in \operatorname{Mat}_{n+1}(\Bbbk)$ is a *symmetric* square matrix. Every non-diagonal element $a_{ij} = a_{ji}$ of A equals the half[1] of coefficient of monomial $x_i x_j$ in the reduced expansion for q. The matrix A is called *the Gram matrix* of q in the chosen basis of V.

In other words, for any quadratic polynomial q on V, there exists a unique symmetric bilinear form $\widetilde{q} : V \times V \to \Bbbk$ such that $q(v) = \widetilde{q}(v, v)$ for all $v \in V$. In coordinates,

$$\widetilde{q}(x, y) = \sum a_{ij} x_i y_j = xAy^t = \frac{1}{2} \sum y_i \frac{\partial q(x)}{\partial x_i}\,. \tag{2-1}$$

In coordinate-free terms, $\widetilde{q}(x, y) = \frac{1}{2}\big(q(x + y) - q(x) - q(y)\big) = \frac{1}{4}\big(q(x + y) - q(x - y)\big)\,.$

Exercise 2.1. Check this.

The symmetric bilinear form $\widetilde{q}$ is called *the polarization* of quadratic form q. It can be thought of as an inner product on V, possibly degenerated. The elements of Gram matrix equal the inner products of basic vectors: $a_{ij} = \widetilde{q}(e_i, e_j)$. In the matrix notations, $A = e^t \cdot e$, where $e = (e_0, e_1, \dots, e_n)$ is the row of basic vectors in V, e^t is the transposed column of basic vectors, and $u \cdot w \overset{\text{def}}{=} \widetilde{q}(u, w) \in \Bbbk$ for $u, w \in V$. When we pass to another basis $e' = eC$, where $C \in \mathrm{GL}_{n+1}(\Bbbk)$, the Gram matrix A of e is related with the Gram matrix A' of e' as $A' = C^t AC$, because $(e')^t \cdot e' = C^t e^t \cdot eC$.

2.1.2 The Gram dterminant. Since $\det A' = \det A \cdot \det^2 C$, the determinant of Gram matrix does not depend on the choice of basis up to multiplication by non zero squares from $\Bbbk$. We write $\det q \in \Bbbk/\Bbbk^{*2}$ for the class of $\det A$ modulo multiplication by non zero squares, and call it *the Gram determinant* of quadratic form $q \in S^2V^*$. The form q and quadric $Q = V(q)$ are called *smooth* or *non-singular*, if $\det q \neq 0$. Otherwise they are called *singular* or *degenerated*.

2.1.3 The rank. Since the rank of matrix is not changed under multiplications of the matrix by non-degenerated matrices, the rank of Gram matrix does not depend on the choice of basis as well. It is called *the rang* of quadretic form q and quadric $Q = V(q)$, and denoted by $\operatorname{rk} q = \operatorname{rk} Q \overset{\text{def}}{=} \operatorname{rk} A$.

Proposition 2.1 (Lagrange's theorem)
For any quadratic form q there exists a basis where the Gram matrix of q is diagonal.

Proof. Induction on $\dim V$. If $q \equiv 0$ or $\dim V = 1$, then the Gram matrix is diagonal. If $\dim V \geqslant 2$ and $q(e) = \widetilde{q}(e, e) \neq 0$ for some $e \in V$, we put $e_1 = e$ to be the first vector of desired basis. Every vector $v \in V$ admits a unique decomposition $v = \lambda e + u$, where $\lambda \in \Bbbk$ and $u \in v^\perp = \{\, w \in V \mid \widetilde{q}(v, w) = 0 \,\}$. Indeed, the orthogonality of v and $v - \lambda e$ forces $\lambda = \widetilde{q}(e, v)/\widetilde{q}(e, e)$, then $u = v - (\widetilde{q}(e, v)/\widetilde{q}(e, e)) \cdot e$.

Exercise 2.2. Verify that $v - \big(\widetilde{q}(e, v)/\widetilde{q}(e, e)\big) \cdot e \in e^\perp$.

Thus, we have the orthogonal decomposition $V = \Bbbk \cdot e \oplus e^\perp$. By induction, there exists a basis $e_2, \dots, e_n$ in $e^\perp$ with diagonal Gram matrix. Hence, $e_1, e_2, \dots, e_n$ is a required basis for V. □

[1] Note that if $\operatorname{char}\Bbbk = 2$, such the matrix A does not always exists.

Corollary 2.1
Every quadratic form q over an algebraically closed field turns to the sum of squares

$$q(x) = x_0^2 + x_1^2 + \cdots + x_k^2 , \quad \text{where } k+1 = \operatorname{rk} q \,,$$

in appropriate coordinates on V.

Proof. Pass to a basis $e_0, e_1, \dots, e_n$ in which the Gram matrix is diagonal, renumber the vectors e_i in order to have $q(e_i) \neq 0$ exactly for $1 \leqslant i \leqslant k$, then multiply all these e_i by $1/\sqrt{q(e_i)} \in \Bbbk$. □

Example 2.1 (quadrics on $\mathbb{P}_1$)
It follows from the Proposition 2.1 that the equation of any quadric $Q \subset \mathbb{P}_1$ can be written in appropriate coordinates on $\mathbb{P}_1$ either as $x_0^2 = 0$ or as $x_0^2 + a x_1^2 = 0$, where $a \neq 0$. In the first case, Q is singular, $\operatorname{rk} Q = 1$, and the equation of Q is the squared linear equation of the point $(0 : 1)$. By this reason, such a quadric is called a *double point*. In the second case, $\operatorname{rk} Q = 2$, the quadric is smooth, and its Gram determinant equals a up to multiplication by non-zero squares. If $-a \in \Bbbk$ is not a square, then the equation $(x_0 / x_1)^2 = -a$ has no solutions, and the quadric is empty. If $-a = \delta^2$ for some $\delta \in \Bbbk$, then $x_0^2 + a x_1^2 = (x_0 - \delta x_1)(x_0 + \delta x_1)$ has two distinct roots $(\pm\delta : 1) \in \mathbb{P}_1$. Thus, the geometry of quadric $Q = V(q) \subset \mathbb{P}_1$ is completely determined by the Gram determinant $\det q \in \Bbbk/(\Bbbk^*)^2$. If $\det q = 0$, then the quadric is a double point. If $-\det q = 1$, that is, $-\det A \in \Bbbk$ is a square, then the quadric consists of two distinct points. If $-\det q \neq 1$, that is, $-\det A \in \Bbbk$ is not a square, then the quadric is empty. Note that the latter case never appears over an algebraically closed field $\Bbbk$.

2.2 Tangent lines. It follows from the Example 2.1 that there are precisely 4 different positional relationships between a quadric Q and a line ℓ in $\mathbb{P}_n$: either $\ell \subset Q$, or $\ell \cap Q$ is a double point, or $\ell \cap Q$ is a pair of distinct points, or $\ell \cap Q = \varnothing$, and the latter case never appears over an algebraically closed field.

Definition 2.1 (tangent space of quadric)
A line ℓ is called *tangent* to a quadric Q at a point $p \in Q$, if either $p \in \ell \subset Q$ or $Q \cap \ell$ is the double point p. In these cases we say that ℓ *touches* Q at p. The union of all tangent lines touching Q at a given point $p \in Q$ is called *the tangent space* to Q at p and denoted by $T_p Q$.

Proposition 2.2
A line (ab) touches a quadric $Q = V(q)$ at the point $a \in Q$ if and only if $\tilde{q}(a, b) = 0$.

Proof. The Gram matrix of restriction $q|_{(a,b)}$ in the basis a, b of line (ab) is

$$\begin{pmatrix} \tilde{q}(a,a) & \tilde{q}(a,b) \\ \tilde{q}(a,b) & \tilde{q}(b,b) \end{pmatrix} .$$

Since $\tilde{q}(a,a) = q(a) = 0$ by assumption, the Gram determinant $\det q|_{(a,b)} = \tilde{q}(a,b)^2$. It vanishes if and only if $\tilde{q}(a,b) = 0$. □

Corollary 2.2 (apparent contour of quadric)
For any point $p \notin Q$, the *apparent contour* of Q viewed from p, i.e., the set of all points $a \in Q$ such that the line (pa) touches Q at a, is cut out Q by the hyperplane $\Pi_p \overset{\text{def}}{=} \{x \in \mathbb{P}_n \mid \tilde{q}(p, x) = 0\}$.

Proof. Since $\tilde{q}(p,p) = q(p) \neq 0$, the equation $\tilde{q}(p,x) = 0$ is a non-trivial linear homogeneous equation on x. Thus, $\Pi_p \subset \mathbb{P}_n$ is a hyperplane, and $Q \cap \Pi$ coincides with the apparent contour of Q viewed from p by the Proposition 2.2. □

2.2.1 Smooth and singular points. Associated with a quadratic form $q \in S^2V^*$ is the linear mapping

$$\widehat{q} : V \to V^*, \quad v \mapsto \widetilde{q}(\,*\,, v)\,, \tag{2-2}$$

sending a vector $v \in V$ to the linear form $\widehat{q}(v)\colon V \to \Bbbk$, $w \mapsto \widetilde{q}(w, v)$. The map (2-2) is called *the correlation* of quadratic form q.

EXERCISE 2.3. Convince yourself that the matrix of linear map (2-2) written in dual bases e, x of V and V^* coincides with the Gram matrix of q in the basis e.

This shows once more, that the rank $\operatorname{rk} A = \dim V - \dim\ker\widehat{q}$ does not depend on a choice of basis. The vector space $\ker(q) \overset{\text{def}}{=} \ker\widehat{q} = \{\, v \in V \mid \widetilde{q}(w, v) = 0 \ \ \forall\, w \in V \,\}$ is called *the kernel* of quadratic form q. The projectivization of the kernel is denoted

$$\operatorname{Sing} Q \overset{\text{def}}{=} \mathbb{P}(\ker q) = \{p \in \mathbb{P}(V) \mid \forall\, u \in V \ \widehat{q}(p, u) = 0\}$$

and called *the vertex space* or *the singular locus* of quadric $Q = V(q) \subset \mathbb{P}_n$. The points of $\operatorname{Sing} Q$ are called *singular*. All points of the complement $Q \smallsetminus \operatorname{Sing} Q$ are called *smooth*. Thus, a point $p \in Q \subset \mathbb{P}(V)$ is smooth if and only if the tangent space $T_pQ = \{x \in \mathbb{P}_n \mid \widetilde{q}(p, x) = 0\}$ is a hyperplane in $\mathbb{P}_n$. Conversely, a point $p \in Q \subset \mathbb{P}(V)$ is singular if and only if the tangent space $T_pQ = \mathbb{P}(V)$ is the whole space, that is, any line passing through a either lies on Q or does not intersect Q anywhere besides a.

EXERCISE 2.4. Convince yourself that the singularity of a point $p \in Q \subset \mathbb{P}_n$ means that

$$\frac{\partial q}{\partial x_i}(p) = 0 \quad \text{for all } 0 \leqslant i \leqslant n.$$

Note that a quadric is smooth in the sense of n° 2.1.2 if and only if it has no singular points.

LEMMA 2.1
If a quadric $Q \subset \mathbb{P}_n$ has a smooth point $a \in Q$, then Q is not contained in a hyperplane.

PROOF. For $n = 1$, this follows from the Example 2.1. Consider $n \geqslant 2$. If Q lies inside a hyperplane H, then every line $\ell \not\subset H$ passing through a intersects Q only in a and therefore is tangent to Q at a. Hence, $\mathbb{P}_n = H \cup T_pQ$. This contradicts to the Exercise 2.5 below. □

EXERCISE 2.5. Show that the projective space over a field of characteristic $\neq 2$ is not a union of two hyperplanes.

THEOREM 2.1
For any quadric $Q \subset \mathbb{P}(V)$ and projective subspace $L \subset \mathbb{P}(V)$ complementary to $\operatorname{Sing} Q$, the intersection $Q' = L \cap Q$ is a smooth quadric in L, and Q is the linear join[1] of Q' and $\operatorname{Sing} Q$.

PROOF. Let $L = \mathbb{P}(U)$. Then $V = \ker q \oplus U$. Assume that there exists a vector $u \in U$ such that $\widetilde{q}(u, u') = 0$ for all $u' \in U$. Since $\widetilde{q}(u, w) = 0$ for all $w \in \ker q$ as well, the equality $\widetilde{q}(u, v) = 0$ holds for all $v \in V$. Hence, $u \in \ker q \cap U = 0$. That is, Q' is smooth. Every line ℓ that intersects $\operatorname{Sing} Q$ but is not contained in $\operatorname{Sing} Q$ does intersect L and either is contained in Q or does not intersect Q anywhere besides the point $\ell \cap \operatorname{Sing} Q$. This forces Q to be the union of lines (sp) such that $s \in \operatorname{Sing} Q$, $p \in L \cap Q$. □

[1]For sets $X, Y \subset \mathbb{P}_n$, their *linear join* is the union of all lines (xy) such that $x \in X$, $y \in Y$.

2.3 Duality. Projective spaces $\mathbb{P}_n = \mathbb{P}(V)$, $\mathbb{P}_n^\times \stackrel{\text{def}}{=} \mathbb{P}(V^*)$, obtained from dual vector spaces V, V^*, are called *dual*. Geometrically, $\mathbb{P}_n^\times$ is the space of hyperplanes in $\mathbb{P}_n$, and vice versa. The linear equation $\langle\, \xi\,,\, v\,\rangle = 0$, being considered as an equation on $v \in V$ for a fixed $\xi \in V^*$, defines a hyperplane $\mathbb{P}(\operatorname{Ann}\xi) \subset \mathbb{P}_n$. As an equation on ξ for a fixed v, it defines a hyperplane in $\mathbb{P}_n^\times$ formed by all hyperplanes in $\mathbb{P}_n$ passing through v. For every $k = 0, 1, \dots, n$ there is the canonical involutive[1] bijection $L \leftrightarrow \operatorname{Ann} L$ between projective subspaces of dimension k in $\mathbb{P}_n$ and projective subspaces of dimension $(n-k-1)$ in $\mathbb{P}_n^\times$. It is called *the projective duality*. For a given $L = \mathbb{P}(U) \subset \mathbb{P}_n$, the dual subspace $\operatorname{Ann} L \stackrel{\text{def}}{=} \mathbb{P}(\operatorname{Ann} U) \subset \mathbb{P}_n^\times$ consists of all hyperplanes in $\mathbb{P}_n$ containing L. The projective duality reverses inclusions: $L \subset H \iff \operatorname{Ann} L \supset \operatorname{Ann} H$, and sends intersections to linear joins, and vise versa. This allows to translate the theorems true for $\mathbb{P}_n$ to the dual statements about the dual figures in $\mathbb{P}_n^\times$. The latter may look quite dissimilar to the original. For example, the collinearity of 3 points in $\mathbb{P}_n$ is translated as the existence of codimension-2 subspace common for 3 hyperplanes in $\mathbb{P}_n^\times$.

2.3.1 The polar mapping. For a smooth quadric $Q = V(q)$, the correlation $\widehat{q} : V \to V^*$ is an isomorphism. The induced linear projective isomorphism $\overline{q} : \mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V^*)$ is called the *polar mapping* or *the polarity* provided by quadric Q. The polarity sends a point $p \in \mathbb{P}_n$ to the hyperplane

$$\varPi_p = \operatorname{Ann} \overline{q}(p) = \{ x \in \mathbb{P}(V) \mid \widetilde{q}(p, x) = 0 \}\,,$$

which cuts apparent contour of Q viewed from p in accordance with the Corollary 2.2. The hyperplane $\varPi_p$ and point p are called *the polar* and *pole* of one other with respect to Q. If $p \in Q$, then $\varPi_p = T_p Q$ is the tangent plane to Q at p. Note that a lies on the polar of b if and only if b lies on the polar of a, because the condition $\widetilde{q}(a, b) = 0$ is symmetric. Such points a, b are called *conjugated* with respect to the quadric $Q = V(q)$.

Proposition 2.3
Let a line (ab) intersect a smooth quadric Q in two distinct points c, d different from a, b. Then a, b are conjugated with respect to Q if and only if they are harmonic to c, d.

Proof. Chose some homogeneous coordinate $x = (x_0 : x_1)$ on the line $\ell = (ab) = (cd)$. The intersection $Q \cap \ell = \{c, d\}$ considered as a quadric in ℓ is the zero set of quadratic form

$$q(x) = \det(x, c) \cdot \det(x, d)\,,$$

whose polarization is $\widetilde{q}(x, y) = \frac{1}{2}\big(\det(x, c) \cdot \det(y, d) + \det(y, c) \cdot \det(x, d)\big)$. Thus, $\widetilde{q}(a, b) = 0$ means that $\det(a, c) \cdot \det(b, d) = -\det(b, c) \cdot \det(a, d)$, i.e., $[a, b, c, d] = -1$. □

Proposition 2.4
Let $G, Q \subset \mathbb{P}_n$ be two quadrics with Gram matrices A, $\varGamma$ in some basis of $\mathbb{P}_n$. If G is smooth, then the polar mapping of G sends Q to the quadric $Q_G^\times \subset \mathbb{P}_n^\times$ which has the Gram matrix $A_\varGamma^\times = \varGamma^{-1} A \varGamma^{-1}$ in the dual basis of $\mathbb{P}_n^\times$. Note that $\operatorname{rk} Q_G^\times = \operatorname{rk} Q$.

Proof. Write the homogeneous coordinates in $\mathbb{P}_n$ as row vectors x and dual coordinates in $\mathbb{P}_n^\times$ as column vectors ξ. The polarity $\mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n^\times$ provided by G sends $x \in \mathbb{P}_n$ to $\xi = \varGamma x^t$. Since $\varGamma$ is invertible, x is recovered from ξ as $x = \xi^t \varGamma^{-1}$. When x runs through the quadric $x A x^t = 0$, the corresponding ξ fills the quadric $\xi^t \varGamma^{-1} A \varGamma^{-1} \xi = 0$. □

[1] That is, inverse to itself: $\operatorname{Ann}\operatorname{Ann} L = L$.

Corollary 2.3
The tangent spaces to a smooth quadric $Q \subset \mathbb{P}_n$ form the smooth quadric $Q^\times \subset \mathbb{P}_n^\times$. The Gram matrices of Q, $Q^\times$ in dual bases of $\mathbb{P}_n$, $\mathbb{P}_n^\times$ are inverse to each other.

Proof. Put $G = Q$ and $\Gamma = A$ in the Proposition 2.4. □

2.3.2 Polarities over non-closed fields. If $\Bbbk$ is not algebraically closed, then there are non-singular quadratic forms $q \in S^2V^2$ with $V(q) = \varnothing$. However, their polarities $\overline{q} : \mathbb{P}(V) \to \mathbb{P}(V^*)$, that is, the bijective correspondences between points and hyperplanes, are non-trivial anyway.

Exercise 2.6. Describe geometrically the polarity with respect to «imaginary circle» $x^2+y^2 = -1$ in the Euclidean plane $\mathbb{R}^2$.

Thus, the polarities are much more informative than the quadrics. The quadric is recovered from its polarity as the set of all points lying on the own polars, i.e., the self-conjugated points. It follows from the Theorem 1.1 that two polarities coincide if and only if the corresponding quadratic forms are proportional. Thus, the polarities on $\mathbb{P}_n = \mathbb{P}(V)$ stay in bijection with the points of projective space $\mathbb{P}(S^2V^*) = \mathbb{P}_{\frac{n(n+3)}{2}}$. Somewhat erroneous, the latter is called *the space of quadrics* in $\mathbb{P}(V)$. The quadrics $Q \subset \mathbb{P}_n$ passing through a given point $p \in \mathbb{P}_n$ form a hyperplane in the space of quadrics, because the equation $q(p) = 0$ is linear homogeneous in $q \in \mathbb{P}(S^2V^*)$.

Proposition 2.5
Every collection of $n(n+3)/2$ points in $\mathbb{P}_n$ lies on some quadric.

Proof. Any $n(n+3)/2$ hyperplanes in $\mathbb{P}_{\frac{n(n+3)}{2}}$ have a non empty intersection. □

Proposition 2.6
Over an infinite field, two nonempty smooth quadrics coincide if and only if their equations are proportional.

Proof. If $V(q_1) = V(q_2)$ in $\mathbb{P}(V)$, then two polarities $\overline{q}_1, \overline{q}_2 : \mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V^*)$ coincide in all points of the quadrics. It follows from the Corollary 1.1 on p. 12 and the Exercise 2.7 below that the correlation maps $\widehat{q}_1, \widehat{q}_2 : V \xrightarrow{\sim} V^*$ and therefore the Gram matrices are proportional. □

Exercise 2.7. Check that over an infinite field, every nonempty smooth quadric $\mathbb{P}_n$ contains $n+2$ points such that no $n+1$ of them lie within a hyperplane.

2.4 Conics. Plane quadrics are called *conics*. For $\mathbb{P}_2 = \mathbb{P}(V)$, the space of conics $\mathbb{P}(S^2V^*) = \mathbb{P}_5$. Conics of rank 1 are called *a double lines*. In appropriate coordinates, such a conic has the equation $x_0^2 = 0$. It is totally singular, i.e., has no smooth points at all. By the Theorem 2.1 on p. 18, a conic S of rank 2 is the linear join of the singular point $s \in S$ and a smooth quadric $S \cap \ell$ within a line $\ell \not\ni s$. By the Example 2.1 on p. 17, $S \cap \ell$ either consists of two distinct points or is empty. In the first case, S is the union of two lines intersecting at the singular point s. Such a conic is called *split*. If $S \cap \ell = \varnothing$, then $S = \{s\}$ consists of the singular point only. For example, the conic $x_0^2 + x_1^2 = 0$ in $\mathbb{P}(\mathbb{R}^3)$ is of this sort. Over an algebraically closed field, there are no such conics, certainly.

Lemma 2.2 (rational parametrization of non-empty smooth conic)
Every non-empty smooth conic $C \subset \mathbb{P}_2$ over any field $\Bbbk$ with char $\Bbbk \neq 2$ admits a rational quadratic parametrization, i.e., there exist homogeneous quadratic polynomials $\varphi_0, \varphi_1, \varphi_2 \in \Bbbk[t_0, t_1]$ such that the map $\varphi : \mathbb{P}_1 \to \mathbb{P}_2, (t_0 : t_1) \mapsto \big(\varphi_0(t_0,t_1) : \varphi_1(t_0,t_1) : \varphi_2(t_0,t_1)\big)$, establishes a bijection between $\mathbb{P}_1$ and C.

Proof. Given a point $p \in C$, a required parametrization is provided by the projection $\varphi : \ell \dashrightarrow C$ of an arbitrary line $\ell \not\ni p$ from p onto C. For every $t \in \ell$, the line (pt) intersects C at p and one more point, which coincides with p, if $(pt) = T_pC$, and differs from p for all other t. In the first case we put $\varphi(t) = a$. For all other t, the second intersection point can be written as $t + \lambda p$, where $\lambda \in \Bbbk$, and satisfies the equation $\widetilde{q}(t + \lambda p, t + \lambda p) = 0$, which is equivalent to $q(t) = -2\lambda\widetilde{q}(t, p)$. Thus, the map $\varphi : \ell \dashrightarrow C$ takes $t \in \ell$ to $\varphi(t) = q(t) \cdot p - 2q(p, t) \cdot t \in C$. □

Exercise 2.8. Verify that the right hand side of the latter formula equals p for $t = T_pC \cap \ell$, and make sure that φ is described in coordinates by a triple of quadratic homogeneous polynomials in the coordinates of t as required.

Lemma 2.3
The intersection $C \cap D$ of a smooth conic C with a curve D of degree d in $\mathbb{P}_2$ either consists of at most $2d$ points or coincides with C.

Proof. Let $\varphi : \mathbb{P}_1 \to \mathbb{P}_2$, $(t_0 : t_1) \mapsto \big(\varphi_0(t_0, t_1) : \varphi_1(t_0, t_1) : \varphi_2(t_0, t_1)\big)$ be a rational quadratic parameterization of C, and $D = V(f)$ for some homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d. The values of parameter t corresponding to the intersection point $C \cap D$ satisfy the equation $f\big(\varphi_0(t), \varphi_1(t), \varphi_2(t)\big) = 0$, whose left hand side is either the zero polynomial or a non-zero homogeneous polynomial of degree $2d$. In the first case $C \subset D$. In the second case the equation has at most $2d$ solutions in $\mathbb{P}_1$. □

Proposition 2.7
Any 5 points in $\mathbb{P}_2$ lie on a conic. Such a conic C is unique if and only if every 4 of the points are non-collinear. If every 3 of the points are non-collinear, the conic C is smooth.

Proof. The first statement is exactly the Proposition 2.5 for $n = 2$. Let a line ℓ pass through some 3 of the given points. Then any conic C passing through the given points contains ℓ. If the remaining two pints a, b do not lie on ℓ, then $C = \ell \cup (ab)$ is unique. If $a \in \ell$, then for any line $\ell' \ni b$, the split conic $\ell \cup \ell'$ contains all five given points. If any 3 of the given points are non-collinear, then every conic passing through the 5 given points is smooth, because a singular conic is either a line, or a pair of lines, or a point. Since two different smooth conics have at most 4 intersection points by the Lemma 2.3, a smooth conic passing through 5 points is unique. □

Corollary 2.4
Any 5 lines without triple intersections in $\mathbb{P}_2$ do touch a unique smooth conic.

Proof. This is projectively dual to the last statement in the Proposition 2.7. □

2.5 Quadratic surfaces. The space of quadrics in $\mathbb{P}_3 = \mathbb{P}(V)$ is $\mathbb{P}(S^2V^*) = \mathbb{P}_9$. In particular, any 9 points in $\mathbb{P}_3$ lie on some quadric.

Exercise 2.9. Show that any 3 lines in $\mathbb{P}_3$ lie on a quadric.

A quadratic surface of rank 1 is called *a double plane*. It is totally singular and has the equation $x_0^2 = 0$ in appropriate coordinates on $\mathbb{P}_3$. A quadratic surface S of rang 2 either is *a split quadric*, i.e., a union of two planes intersecting along the singular line $\ell = \operatorname{Sing} S$, or is exhausted by the singular line, and the latter case is impossible over an algebraically closed field.

Exercise 2.10. Prove this.

A quadratic surface $S \subset \mathbb{P}_3$ of rank 3 is called *a simple cone*. It is ruled by the lines (sp), where $s \in S$ is the singular point and t runs through a smooth conic $C = S \cap \Pi$ laying in a plane $\Pi \not\ni s$. Note that C may be empty as soon the ground field is not algebraically closed. In this case $S = \{s\}$ is exhausted by the singular point. If $C \neq \varnothing$, the linear span of C is the whole Π.

EXERCISE 2.11. Convince yourself that the lines laying on a simple cone with vertex s over a smooth conic C are exhausted by the lines (st), $t \in C$.

As a byproduct of the previous discussion, we get

PROPOSITION 2.8
Every 3 mutually non-intersecting lines in $\mathbb{P}_3$ lie on a smooth quadratic surface. □

Over an algebraically closed field, all smooth quadrics in $\mathbb{P}_3$ are congruent modulo the linear projective automorphisms of $\mathbb{P}_3$. The most convenient model of the smooth quadric is described below.

2.5.1 The Segre quadric. Let U be a vector space of dimension 2. Write $W = \mathrm{End}(U)$ for the space of linear maps $F : U \to U$, and consider $\mathbb{P}_3 = \mathbb{P}(W)$. A choice of basis in U identifies W with the space $\mathrm{Mat}_2(\Bbbk)$ of 2×2 matrices. The quadric

$$S \stackrel{\text{def}}{=} \{F :\in \mathrm{End}(U) \mid \det F = 0\} = \left\{ \begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} \;\middle|\; x_0x_3 - x_1x_2 = 0 \right\} \subset \mathbb{P}_3 \tag{2-3}$$

is called *the Segre quadric*. It is formed by endomorphisms of rank 1 considered up to proportionality. The image of an operator $F : U \to U$ of rank 1 has dimension 1 and is spanned by a non zero vector $v \in U$, uniquely determined by F up to proportionality. The value of F on an arbitrary vector $u \in U$ equals $F(u) = \xi(u) \cdot v$, where $\xi \in U^*$ is a linear form such that $\mathrm{Ann}\,\xi = \ker F$. Note that ξ is uniquely determined by F and $v \in \mathrm{im}\,F \smallsetminus 0$. Conversely, for any non-zero $v \in U$, $\xi \in U^*$ the operator

$$\xi \otimes v : U \to U, \quad u \mapsto \xi(u)v$$

has rank 1, its image is spanned by v, and the kernel equals $\mathrm{Ann}\,\xi$. Thus, we have the well defined injective map

$$s : \mathbb{P}(U^*) \times \mathbb{P}(U) \hookrightarrow \mathbb{P}\,\mathrm{End}(U), \quad (\xi, v) \mapsto \xi \otimes v, \tag{2-4}$$

whose image coincides with the Segre quadric (2-3). This map is called *the Segre embedding*.

The rows of any 2×2 matrix of rank 1 are proportional, as well as the columns. The matrices with a fixed ratio ([row 1] : [row 2]) $= (t_0 : t_1)$ or ([column 1] : [column 2]) $= (\xi_0 : \xi_1)$ form a vector subspace of dimension 2 in $W = \mathrm{Mat}_2(\Bbbk)$. After the projectivization these subspaces turns to the two families of lines ruling the Segre quadric. These lines are the images of «coordinate lines» $\mathbb{P}_1^\times \times v$ and $\xi \times \mathbb{P}_1$ on the product $\mathbb{P}_1^\times \times \mathbb{P}_1 = \mathbb{P}(U^*) \times \mathbb{P}(U)$ under the bijection $\mathbb{P}_1^\times \times \mathbb{P}_1 \xrightarrow{\sim} S$ provided by the Segre embedding (2-4). Indeed, the operator $\xi \otimes v$ build from from $\xi = (\xi_0 : \xi_1) \in U^*$ and $v = (t_0 : t_1) \in U$ has the matrix

$$\begin{pmatrix} t_0 \\ t_1 \end{pmatrix} \cdot \begin{pmatrix} \xi_0 & \xi_1 \end{pmatrix} = \begin{pmatrix} \xi_0 t_0 & \xi_1 t_0 \\ \xi_0 t_1 & \xi_1 t_1 \end{pmatrix} \tag{2-5}$$

with the prescribed ratios $(t_0 : t_1)$ and $(\xi_0 : \xi_1)$ between the rows and columns respectively. Since the Segre map $\mathbb{P}_1^\times \times \mathbb{P}_1 \xrightarrow{\sim} S$ is bijective, the incidence relations among coordinate lines in $\mathbb{P}_1^\times \times \mathbb{P}_1$ are the same as among their images in S. That is, within each ruling family, all the lines are mutually non-intersecting, every two lines from different ruling families are intersecting, and each point on the Segre quadric is an intersection point of exactly two lines from different families.

EXERCISE 2.12. Prove that all lines $\ell \subset S$ are exhausted by these two ruling families.

PROPOSITION 2.9 (CONTINUATION OF THE PROPOSITION 2.8)
A smooth quadric Q passing through a triple ℓ_1, ℓ_2, ℓ_3 of mutually non-intersecting lines in $\mathbb{P}_3$, as in the Proposition 2.8, is ruled by all those lines in $\mathbb{P}_3$ that do intersect all the lines ℓ_i. In particular, this quadric is unique.

PROOF. If a line ℓ intersects all the lines ℓ_i, it has at least 3 distinct points on Q and therefore lies on Q. On the other side, for any point $a \in Q$ not laying on the lines ℓ_i, the tangent plane T_aQ intersects every line ℓ_i at some point $p_i \neq a$. Since the line (ap_i) touches Q at a, it lies on Q. Thus, all three lines (ap_i) lie on the conic $Q \cap T_aQ$. Hence, at least two of them, say (ap_1), (ap_2), coincide. If p_3 does not belong to the line $\ell = (ap_1) = (ap_2)$, then the tangent plane $T_{p_3}Q$ intersects ℓ at a point b different from a and all p_i's. The line $(p_3b) \subset Q$ by the same reason as above. Thus, Q contains the triangle abp_3 formed by 3 distinct lines ℓ, (ap_3), and (ab). Hence, Q contains the whole plane spanned by this triangle[1].

EXERCISE 2.13. Show that a smooth quadric in $\mathbb{P}_3$ can not contain a plane.

Therefore, the points a, p_1, p_2, p_3 are collinear, that is, a lies on a line intersecting all the lines ℓ_i. □

EXERCISE 2.14. Given 4 mutually non-intersecting lines in $\mathbb{P}_3$, how many lines intersect them all?

2.6 Linear subspaces lying on a smooth quadric. A smooth quadric Q is called *k-planar*, if there is a projective subspace $L \subset Q$ of dimension $\dim L = k$ and Q does not contain a subspace of higher dimension. By the definition, the planarity of the empty quadric is -1. Thus, the quadrics of planarity 0 are non-empty and do not contain lines.

PROPOSITION 2.10
The planarity of a smooth quadric $Q \subset \mathbb{P}_n$ does not exceed $\dim Q/2 = (n-1)/2$.

PROOF. Let $\mathbb{P}_n = \mathbb{P}(V)$ and $L = \mathbb{P}(W) \subset Q = V(q)$ for some non-singular quadratic form $q \in S^2V^*$ and a vector subspace $W \subset V$. Since $q|_W = 0$, the correlation $\hat{q} : V \xrightarrow{\sim} V^*$ sends W into $\operatorname{Ann}(W)$. Since $\hat{q}$ is injective, $\dim(W) = \dim \hat{q}(W) \leqslant \dim \operatorname{Ann} W = \dim V - \dim W$. Thus, $2\dim W \leqslant \dim V$ and $2\dim L \leqslant n-1$. □

LEMMA 2.4
For any smooth quadric Q and hyperplane Π, the intersection $\Pi \cap Q$ either is a smooth quadric in Π or has exactly one singular point $p \in \Pi \cap Q$. The latter happens if and only if $\Pi = T_pQ$.

PROOF. Let $Q = V(q) \subset \mathbb{P}(V)$, $\Pi = \mathbb{P}(W)$. Since $\dim\ker\left(\hat{q}|_W\right) = \dim\left(W \cap \hat{q}^{-1}(\operatorname{Ann} W)\right) \leqslant$ $\leqslant \dim \hat{q}^{-1}(\operatorname{Ann} W) = \dim \operatorname{Ann} W = \dim V - \dim W = 1$, the quadric $\Pi \cap Q \subset \Pi$ has at most one singular point. If $\operatorname{Sing} Q = \{p\} \neq \varnothing$, then the kernel $\ker \hat{q}|_W \subset W$ has dimension 1 and is spanned by p. Thus, $\operatorname{Ann}(\hat{q}(p)) = W$, that is, $T_pQ = \Pi$. Vice versa, if $\Pi = T_pQ = \mathbb{P}(\operatorname{Ann}\hat{q}(p))$, then $p \in \operatorname{Ann}\hat{q}(p)$ belongs to the kernel of the restriction of $\hat{q}$ on $\operatorname{Ann}\hat{q}$. □

[1] Because for every point of the plane except for the vertexes of triangle, every line passing through this point intersects all three lines ℓ, (ap_3), and (ab).

PROPOSITION 2.11
Let $Q \subset \mathbb{P}_{n+1}$ be a smooth quadric of dimension n. For every $1 \leqslant m \leqslant n/2$, the projective subspaces of dimension m laying in Q and passing through a given point $p \in Q$ stay in bijection with all projective subspaces of dimension $m-1$ laying on a smooth quadric of dimension $n-2$ cut out of Q by any hyperplane $H \subset T_pQ$ complementary to p within the tangent hyperplane $T_pQ \simeq \mathbb{P}_{n-1}$.

PROOF. Every projective subspace $L \subset Q$ of dimension m passing through $p \in Q$ lies inside the intersection $Q \cap T_pQ$, which is the singular quadric in $\mathbb{P}_{n-1} = T_pQ$ with just one singular point p by the Lemma 2.4. It accordance with the Theorem 2.1 on p. 18, the quadric $Q \cap T_pQ \subset \mathbb{P}_{n-1}$ is the cone ruled by lines (pa), where a runs through the smooth quadric Q' cut out of Q by a hyperplane $H \subset \mathbb{P}_{n-1}$ not passing through p. Thus, the subspaces $L \subset Q \cap T_pQ$ of dimension n are exactly the linear joins of p with the subspaces $L' = L \cap H = L \cap Q'$ of dimension $m-1$ laying on Q'. □

COROLLARY 2.5
For any two distinct points a, b on a smooth quadric Q and all $0 \leqslant m \leqslant \dim Q/2$ there is a bijection between the subspaces of dimension m laying on Q and passing through the points a and b respectively. In particular, a projective subspace of dimension k laying on a smooth k-planar quadric can be drown through every point of the quadric.

PROOF. If $b \notin T_aQ$, then $H = T_aQ \cap T_bQ$ does not pass through a, b and lies in the both tangent spaces T_aQ, T_bQ as a hyperplane. By the Proposition 2.11, the sets of projective subspaces $L \subset Q$ of dimension m passing through a and b respectively both stay in bijection with the subspaces $L' \subset Q \cap H$ of dimension $m-1$. If $b \in T_aQ$, pick up a point $c \in Q \smallsetminus (T_aQ \cup T_bQ)$ and repeat the previous arguments twice for the pairs a, c and c, b. □

COROLLARY 2.6
A smooth quadric of dimension n over an algebraically closed field is $[n/2]$-planar.

PROOF. This holds for $n = 0, 1, 2$. Then we use the Proposition 2.11 and induction in n. □

Comments to some exercises

Exrc. 2.4. This follows from the last representation from formula (2-1) on p. 16.

Exrc. 2.5. Let $\mathbb{P}(V) = \mathbb{P}(\operatorname{Ann}\xi) \cup \mathbb{P}(\operatorname{Ann}\eta)$ for some non zero covectors $\xi, \eta \in V^*$. Then the quadratic form $q(v) = \xi(v)\eta(v)$ vanishes identically on V. Therefore its polarization $\widetilde{q}(u,w) = (q(u+w) - q(u) - q(w))/2$ also vanishes. Hence, the Gram matrix of q equals zero, i.e., q is the zero polynomial. However, the polynomial ring has no zero divisors.

Exrc. 2.7. Use the Lemma 2.1 on p. 18 and prove that non-empty smooth quadric over an infinite field can not be covered by a finite number of hyperplanes.

Exrc. 2.9. Pick up some 3 on each line and draw a quadric through these 9 points.

Exrc. 2.10. By the Theorem 2.1 on p. 18, S is the linear join of the singular line $\operatorname{Sing} S$ and a smooth quadric $S \cap \ell$ within a line ℓ complementary to $\operatorname{Sing} S$. This smooth quadric is either a pair of distinct points or empty.

Exrc. 2.12. Every line $\ell \subset S$ passing through a given point $a \in S$ lies inside $S \cap T_a S$, which is the split conic exhausted by two ruling lines crossing at a.

Exrc. 2.13. See the Proposition 2.10 on p. 23.

Exrc. 2.14. Use *the method of loci*: remove one of the given lines and look how does the locus filled by the lines crossing 3 remaining lines interact with the removed line.