§1 Projective geometry

1.1 Preliminaries. Algebraic geometry deals with figures looking locally[1] as a set of solutions for some system of polynomial equations on affine space. Recall briefly what does the latter mean.

1.1.1 Polynomials. Let V be a vector space of dimension n over a field $\Bbbk$. Its *dual space* V^* is the space of all linear maps $V \to \Bbbk$, also known as *linear forms* or *covectors*. We write $\langle\, \varphi\,,\, v\,\rangle = \varphi(v) \in \Bbbk$ for the value of a covector $\varphi \in V^*$ on a vector $v \in V$. Given a basis $e_1, e_2, \dots, e_n \in V$, its *dual basis* $x_1, x_2, \dots, x_n \in V^*$ consists of the coordinate linear forms defined by prescriptions

$$\langle\, x_i\,,\, e_j\,\rangle = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

We write $SV^* = \Bbbk[x_1, x_2, \dots, x_n]$ for the algebra of polynomials in x_i's with coefficients in $\Bbbk$. Another choice of basis in V^* leads to an isomorphic algebra whose generators are obtained from x_i's by invertible linear change of variables. We write $S^dV^* \subset SV^*$ for the subspace of homogeneous polynomials of degree d. This subspace is not changed under linear changes of variables. A basis of S^dV^* is formed by the monomials $x_1^{m_1}x_2^{m_2}\dots x_n^{m_n}$ numbered by the collections $m = (m_1, m_2, \dots, m_n)$ of integers $0 \leqslant m_i \leqslant d$ such that $\sum m_i = d$.

Exercise 1.1. Make sure that $\dim S^dV^* = \binom{n+d-1}{d}$ as soon $\dim V = n$.

Remark 1.1. Actually, the *symmetric algebra* SV^* and *symmetric powers* S^dV^* of a vector space V^* admit an intrinsic coordinate-free definition, see n° 4.3.1 on p. 44 below. The algebra SV^* is *graded*, i.e.,

$$SV^* = \bigoplus_{d\geqslant 0} S^dV^*$$

as a vector space and $S^kV^* \cdot S^mV^* \subset S^{k+m}V^*$.

1.1.2 Affine space and polynomial functions. Associated with a vector space V of dimension n is the *affine space* $\mathbb{A}^n = \mathbb{A}(V)$, also called the *affinization* of V. By the definition, the points of $\mathbb{A}(V)$ are the vectors of V. The point corresponding to the zero vector is called the *origin* and denoted O. All the other points can be imagined as the heads of non zero radius-vectors drawn from the origin. Every polynomial $f = \sum_m a_m x_1^{m_1}\dots x_n^{m_n} \in SV^*$ produces the *polynomial function*

$$f : \mathbb{A}(V) \to \Bbbk\,, \quad v \mapsto \sum_m a_m \langle\, x_1\,,\, v\,\rangle^{m_1} \dots \langle\, x_n\,,\, v\,\rangle^{m_n}, \tag{1-1}$$

which evaluates the polynomial at the coordinates of points $v \in \mathbb{A}(V)$. Despite the Proposition 1.1 below, this function is traditionally denoted by the same letter as polynomial.

Proposition 1.1
The homomorphism of algebras $\varepsilon : \Bbbk[x_1, x_2, \dots, x_n] \to \{\text{functions } \mathbb{A}^n \to \Bbbk\}$, which sends a polynomial $f \in \Bbbk[x_1, x_2, \dots, x_n]$ to the corresponding polynomial function $f : \mathbb{A}^n \to \Bbbk$, is injective if and only if the ground field $\Bbbk$ is infinite.

[1] That is, in some neighbor of every point.

Proof. If $\Bbbk$ consists of q elements, then the space of all functions $\mathbb{A}^n \to \Bbbk$ consists of q^{q^n} elements whereas the polynomial algebra $\Bbbk[x_1, x_2, \dots, x_n]$ is an infinite set. Hence, homomorphism ε is not injective. Let $\Bbbk$ be infinite. For $n = 1$, any non zero polynomial $f \in \Bbbk[x_1]$ has at most $\deg f$ roots. Hence, the corresponding polynomial function $f : \mathbb{A}^1 \to \Bbbk$ is not the zero function. For $n > 1$, we proceed inductively. Expand $f \in \Bbbk[x_1, x_2, \dots, x_n]$ as[1] $f(x_1, \dots, x_n) = \sum_k f_\nu(x_1, \dots, x_{n-1}) \cdot x_n^k$. If the polynomial function $f : \mathbb{A}^n \to \Bbbk$ vanishes identically, then the evaluation of all coefficients f_k at any point $w \in \mathbb{A}^{n-1} \subset \mathbb{A}^n$ turns f into polynomial $f(w, x_n) \in \Bbbk[x_n]$ that produces the zero function on line $\mathbb{A}^1 \subset \mathbb{A}^n$ passing through w parallel to x_n-axis. Hence, $f(w, x_n) = 0$ in $\Bbbk[x_n]$, i.e., all the coefficients $f_k(w)$ are identically zero functions of $w \in \mathbb{A}^{n-1}$. By induction, they all are the zero polynomials. □

Exercise 1.2. Let p be a prime number, $\mathbb{F}_p = \mathbb{Z}/(p)$ the residue field modulo p. Give an explicit example of non-zero polynomial $f \in \mathbb{F}_p[x]$ that produces the zero function $f : \mathbb{F}_p \to \mathbb{F}_p$.

1.1.3 Affine algebraic varieties. For a polynomial $f \in SV^*$, the set of all zeros of the corresponding polynomial function $f : \mathbb{A}(V) \to \Bbbk$ is denoted $V(f) \stackrel{\text{def}}{=} \{p \in \mathbb{A}(V) \mid f(p) = 0\}$ and called *an affine algebraic hypersurface*. An intersection of affine hypersurfaces is called *an affine algebraic variety*. Thus, an algebraic variety is a figure $X \subset \mathbb{A}^n$ defined by an arbitrary system of polynomial equations. The simplest example of a hypersurface is an *affine hyperplane* given by linear equation $\varphi(v) = c$, where $\varphi \in V^*$ is a non-zero linear form, and $c \in \Bbbk$. Such a hyperplane passes through the origin if and only if $c = 0$. In this case the hyperplane coincides with the affinization $\mathbb{A}(\operatorname{Ann}\varphi)$ of the vector subspace $\operatorname{Ann}(\varphi) = \{v \in V \mid \varphi(v) = 0\}$, annihilated by the covector φ. In general case, an affine hyperplane $\varphi(v) = c$ is the shift of $\mathbb{A}(\operatorname{Ann}\varphi)$ by an arbitrary vector u such that $\varphi(u) = c$.

1.2 Projective space. Much more interesting geometric object associated with a vector space V is the *projective space* $\mathbb{P}(V)$, also called the *projectivization* of V. By the definition, the points of $\mathbb{P}(V)$ are the vector subspaces of dimension one in V or, equivalently, the lines in $\mathbb{A}(V)$ passing through the origin. To see them as «usual dots» we have to intersect these lines with a screen, an affine hyperplane non-passing through the origin, like on fig. 1◇1. We write U_ξ for such the hyperplane given by linear equation $\xi(v) = 1$, where $\xi \in V^* \smallsetminus 0$, and call it the *affine chart* provided by covector ξ.

Exercise 1.3. Convince yourself that the map $\xi \mapsto U_\xi$ establishes a bijection between the non zero covectors and affine hyperplanes in $\mathbb{A}(V)$ that do not pass through the origin.

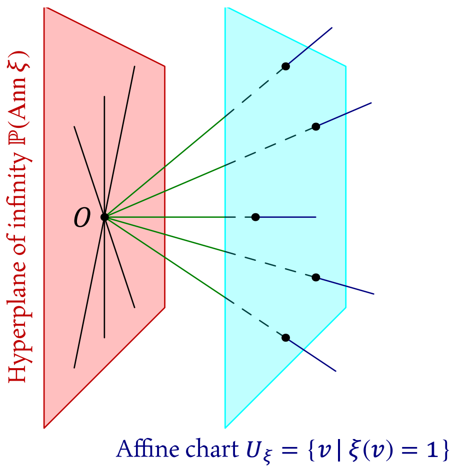

Fig. 1◇1. Projective word.

No affine chart covers the whole $\mathbb{P}(V)$. The difference $\mathbb{P}(V) \smallsetminus U_\xi = \mathbb{P}(\operatorname{Ann}\xi)$ consists of all lines annihilated by ξ, i.e., laying inside the parallel copy of U_ξ drawn through the origin. The projective space formed by these lines is called the *infinity* of affine chart U_ξ.

Every point of $\mathbb{P}(V)$ is covered by some affine chart. For $\dim V = n + 1$, the charts are affine spaces of dimension n, and $\mathbb{P}(V)$ is looking locally as $\mathbb{A}^n$. By this reason, we say that $\mathbb{P}(V)$ has

[1]That is, as a polynomial in x_n with coefficients in the ring $\Bbbk[x_1, x_2, \dots, x_{n-1}]$

dimension n if $\dim V = n+1$, and write $\mathbb{P}_n$ instead of $\mathbb{P}(V)$ when the nature of V is not essential. Note that in a contrast with $\mathbb{A}^n = \mathbb{A}^1 \times \dots \times \mathbb{A}^1$, the space $\mathbb{P}_n$ is not a direct product of n copies of $\mathbb{P}_1$. It follows from fig. 1◇1 that $\mathbb{P}_n = \mathbb{A}^n \sqcup \mathbb{P}_{n-1}$ (a disjoint union). If we repeat this for $\mathbb{P}_{n-1}$ and further, we get the decomposition $\mathbb{P}_n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{P}_{n-2} = \dots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \ldots \sqcup \mathbb{A}^0$, where $\mathbb{A}^0 = \mathbb{P}_0$ is the one point set.

EXERCISE 1.4. Consider this decomposition over the finite field $\mathbb{F}_q$ of q elements and compute the cardinalities of both sides independently. Do you recognize the obtained identity on q?

1.2.1 Homogeneous coordinates. A choice of basis $\xi_0, \xi_1, \dots, \xi_n \in V^*$ identifies V with $\mathbb{k}^{n+1}$ by sending $v \in V$ to $(\xi_0(v), \xi_1(v), \dots, \xi_n(v)) \in \mathbb{k}^{n+1}$. Two coordinate rows $(x_0, x_1, \dots, x_n)$ and $(y_0, y_1, \dots, y_n)$ represent the same point $p \in \mathbb{P}(V)$ if and only if they are proportional, i.e., $x_\mu : x_\nu = y_\mu : y_\nu$ for all $0 \leqslant \mu \neq \nu \leqslant n$, where the identities of type $0 : x = 0 : y$ and $x : 0 = y : 0$ are allowed as well. Thus, the points $p \in \mathbb{P}(V)$ stay in bijection with the collections of ratios $(x_0 : x_1 : \dots : x_n)$. The latter are called *homogeneous coordinates* on $\mathbb{P}(V)$ with respect to the chosen basis.

1.2.2 Local affine coordinates. Pick an affine chart $U_\xi = \{v \in V \mid \xi(v) = 1\}$ on $\mathbb{P}_n = \mathbb{P}(V)$. Any n covectors $\xi_1, \xi_2, \dots, \xi_n \in V^*$ such that $\xi, \xi_1, \xi_2, \dots, \xi_n$ form a basis of V^* provide U_ξ with *local affine coordinates*. Namely, consider the basis $e_0, e_1, \dots, e_m$ in V dual to $\xi, \xi_1, \xi_2, \dots, \xi_n$, and the affine coordinate system with origin at $e_0 \in U_\xi$ and axes $e_1, e_2, \dots, e_n \in \operatorname{Ann} \xi$. The affine coordinates of a point $p \in \mathbb{P}_n$ in this system are computed as follows: rescale p to get the vector $u_p = p/\xi(p) \in U_\xi$ and evaluate n linear forms ξ_ν, $1 \leqslant \nu \leqslant n$, at this vector. The resulting numbers $(t_1(p), t_2(p), \dots, t_n(p))$, where $t_i(p) = \xi_i(u_p) = \xi_i(p)/\xi(p)$ are called *local affine coordinates* of p in the chart U_ξ with respect to the covectors ξ_i. Note that local affine coordinates are *non-linear* functions of homogeneous coordinates.

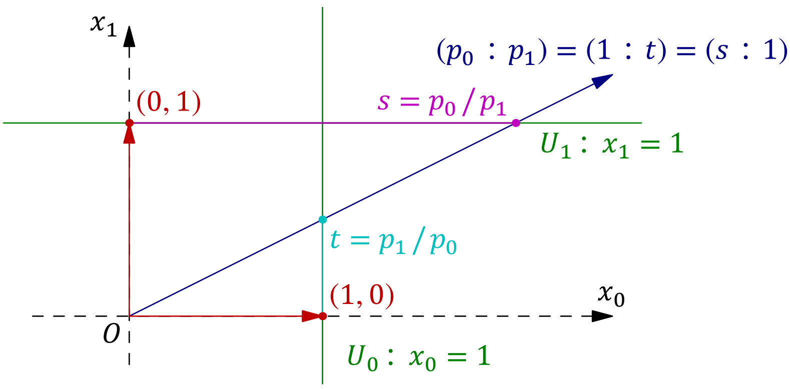

Fig. 1◇2. The standard affine charts on $\mathbb{P}_1$.

EXAMPLE 1.1 (PROJECTIVE LINE)
The projective line $\mathbb{P}_1 = \mathbb{P}(\mathbb{k}^2)$ is covered by two affine charts $U_0 = U_{x_0}$ and $U_1 = U_{x_1}$ represented by the affine lines $x_0 = 1$ and $x_1 = 1$ in $\mathbb{A}^2 = \mathbb{A}(\mathbb{k}^2)$, see fig. 1◇2. The chart U_0 covers the whole $\mathbb{P}_1$ except for the point $(0 : 1)$, the vertical axis in $\mathbb{k}^2$. The function $t = x_1|_{U_0} = x_1/x_0$ can be taken as a local affine coordinate in U_0. The infinite point of the chart U_1 is $(1 : 0)$, the horizontal axis in $\mathbb{k}^2$. The function $s = x_0|_{U_1} = x_0/x_1$ can be taken as a local affine coordinate in U_1. If a point $p = (p_0 : p_1) = (1 : p_1/p_0) = (p_0/p_1 : 1)$ is visible in the both charts, then its coordinates $t = p_1/p_0$ and $s = p_0/p_1$ are inverse to one other. Thus, $\mathbb{P}_1$ is obtained by gluing two distinct

copies of $\mathbb{A}^1 = \Bbbk$ along the complements to zero by the rule: a point s of the first $\mathbb{A}^1$ is identified with the point $1/s$ of the second. Over the field $\mathbb{R}$ of real numbers, this gluing procedure can be visualized as follows. Consider the circle of diameter one and identify two copies of $\mathbb{R}$ with two tangent lines passing through a pair of opposite points of the circle, see fig. 1⋄3. Then map each line to the circle via the central projection from the point opposite to the point of contact. It is immediate from fig. 1⋄3 that $1 : s = t : 1$ for any two points s, t of different lines mapped to the same point of the circle.

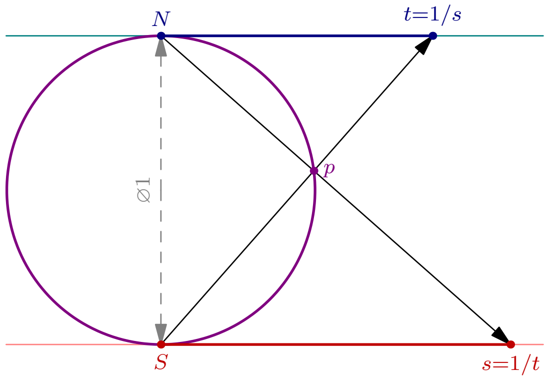

Fig. 1⋄3. $\mathbb{P}_1(\mathbb{R}) \simeq S^1$.

The same construction works for the field $\mathbb{C}$ of complex numbers as well, see fig. 1⋄4. Consider the sphere of diameter one and identify two copies of $\mathbb{C}$ with two tangent planes drown through the south and north poles of the sphere in the way[1] shown on fig. 1⋄4. The central projection of each plane to the sphere from the pole opposite to the point of contact sends complex numbers s, t, laying on different planes, to the same point of sphere if and only if s and t have opposite arguments and inverse absolute values[2], i.e., $t = 1/s$. Thus, the complex projective line can be thought of as the sphere.

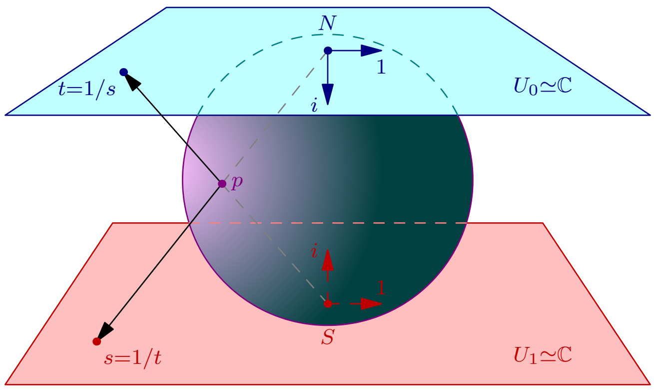

Fig. 1⋄4. $\mathbb{P}_1(\mathbb{C}) \simeq S^2$.

[1]Note that the both planes have compatible orientatons with respect to the sphere in the sense that they can be obtained one from the other by continuous move along the surface of sphere.

[2]The latter follows from fig. 1⋄3.

Exercise 1.5. Make sure that a) the real projective plane $\mathbb{P}_2(\mathbb{R})$ can be obtained by gluing a Möbius tape with a disc along their boundary circles[1] b) the real projective 3D space $\mathbb{P}_3 = \mathbb{P}(\mathbb{R}^4)$ can be identified with the Lie group $\mathrm{SO}_3(\mathbb{R})$ of rotations of the Euclidean space $\mathbb{R}^3$ about the origin.

Example 1.2 (standard affine covering for $\mathbb{P}_n$)
The standard affine covering of $\mathbb{P}_n = \mathbb{P}\left(\Bbbk^{n+1}\right)$ is formed by $n+1$ affine charts $U_\nu \overset{\text{def}}{=} U_{x_\nu} \subset \Bbbk^{n+1}$ given by equations $x_\nu = 1$. For every $\nu = 0, 1, \dots, n$, the functions

$$t_i^{(\nu)} = x_i|_{U_\nu} = \frac{x_i}{x_\nu}, \quad 0 \leqslant i \leqslant n\,,\ i \neq \nu\,,$$

are taken as default local affine coordinates inside U_ν. This allows to think of $\mathbb{P}_n$ as the result of gluing $n+1$ distinct copies $U_0, U_1, \dots, U_n$ of affine space $\mathbb{A}^n$ along their actual intersections inside $\mathbb{P}_n$. In terms of homogeneous coordinates $x = (x_0 : x_1 : \dots : x_n)$ on $\mathbb{P}_n$, the intersection $U_\mu \cap U_\nu$ consists of all $x \in \Bbbk^{n+1}$ such that $x_\mu \neq 0$ and $x_\nu \neq 0$. In terms of local affine coordinates inside U_μ and U_ν respectively, this locus is described by inequalities $t_\nu^{(\mu)} \neq 0$ and $t_\mu^{(\nu)} \neq 0$. Two points $t^{(\mu)} \in U_\mu$ and $t^{(\nu)} \in U_\nu$ are glued together in $\mathbb{P}_n$ if and only if $t_\nu^{(\mu)} = 1/t_\mu^{(\nu)}$ and $t_i^{(\mu)} = t_i^{(\nu)}/t_\mu^{(\nu)}$ for $i \neq \mu, \nu$. The right hand sides of these relations are called the *transition functions* from $t^{(\nu)}$ to $t^{(\mu)}$.

1.3 Projective algebraic varieties. Let us fix some basis $x_0, x_1, \dots, x_n$ in V^*. In a contrast with the affine geometry, a non-constant polynomial $f \in \Bbbk[x_0, x_1, \dots, x_n]$ does not produce a well defined function on $\mathbb{P}(V)$ anymore, since typically $f(v) \neq f(\lambda v)$ for non zero $v \in V$ and $\lambda \in \Bbbk$. However, for any *homogeneous* polynomial $f \in S^d V^*$, the zero set $V(f) = \{\, p \in \mathbb{P}(V) \mid f(v) = 0 \,\}$ is still well defined in $\mathbb{P}(V)$, because $f(v) = 0 \iff f(\lambda v) = \lambda^d f(v) = 0$. In other words, for such f, the affine hypersurface $V(f) \subset \mathbb{A}(V)$ is a cone ruled by lines passing through the origin. The set of these lines is also denoted by $V(f) \subset \mathbb{P}(V)$ and called *a projective hypersurface* of degree $d = \deg f$. An intersection of projective hypersurfaces is called *an algebraic projective variety*.

The simplest example of a projective variety is *a projective subspace* $\mathbb{P}(U) \subset \mathbb{P}(V)$, the projectivization of a vector subspace $U \subset V$. It is described by a system of linear homogeneous equations $\varphi(v) = 0$, where φ runs through $\operatorname{Ann} U \subset V^*$. For example, the projectivized linear span of any two non-proportional vectors $a, b \in V$ is denoted $(ab) \subset \mathbb{P}(V)$ and called *a line*. It consists of ll points represented by the vectors $\lambda a + \mu b$, $\lambda, \mu \in \Bbbk$. Alternatively, it is described by the system of linear equations $\xi(x) = 0$, where ξ runs through the subspace $\operatorname{Ann}(a) \cap \operatorname{Ann}(b) \subset V^*$ or, equivalently, through an arbitrary basis of this subspace. The ratio $(\lambda : \mu)$ can be considered as the internal homogeneous coordinate of the point $\lambda a + \mu b$ on the projective line (ab) with respect to the basis a, b.

Exercise 1.6. Show that $\dim K \cap L \geqslant \dim K + \dim L - n$ for any two projective subspaces $K, L \subset \mathbb{P}_n$. In particular, $K \cap L \neq \varnothing$ soon $\dim K + \dim L \geqslant n$. For example, any two lines on $\mathbb{P}_2$ are intersecting.

Example 1.3 (real affine conics)
Consider the real projective plane $\mathbb{P}_2 = \mathbb{P}(\mathbb{R}^3)$ and the curve C defined by homogeneous equation

$$x_0^2 + x_1^2 = x_2^2\,. \tag{1-2}$$

[1] Note that the boundary of a Möbius tape is a circle as well as the boundary of a disc.

In the standard affine chart U_2, where $x_2 = 1$, in the default local affine coordinates $t_0 = x_0/x_2$, $t_1 = x_1/x_2$, the equation (1-2) turns to the equation of circle $t_0^2 + t_1^2 = 1$. In the chart U_1, where $x_1 = 1$, in the coordinates $t_0 = x_0/x_1$, $t_2 = x_2/x_1$, we get the hyperbola $t_2^2 - t_0^2 = 1$. In the «slanted» chart $U_{x_1+x_2}$, where $x_1 + x_2 = 1$, in the coordinates

$$s = x_0|_{U_{x_1+x_2}} = \frac{x_0}{x_1 + x_2}\,, \quad t = (x_2 - x_1)|_{U_{x_1+x_2}} = \frac{x_2 - x_1}{x_2 + x_1}\,,$$

the equation (1-2) turns[1] to the equation of parabola $s^2 = t$. Thus, the affine ellipse, hyperbola, and parabola are just different pieces of the same projective curve C observed in several affine charts. The shape of C in an affine chart $U_\xi \subset \mathbb{P}_2$ is determined by the positional relationship between C and the infinite line $\ell_\infty = V(\xi)$ of the chart U_ξ. The curve C is looking as an ellipse, hyperbola, and parabola as soon ℓ_∞ does not intersect C, touches C at one point, and intersects C in two distinct points respectively, see. fig. 1⋄5.

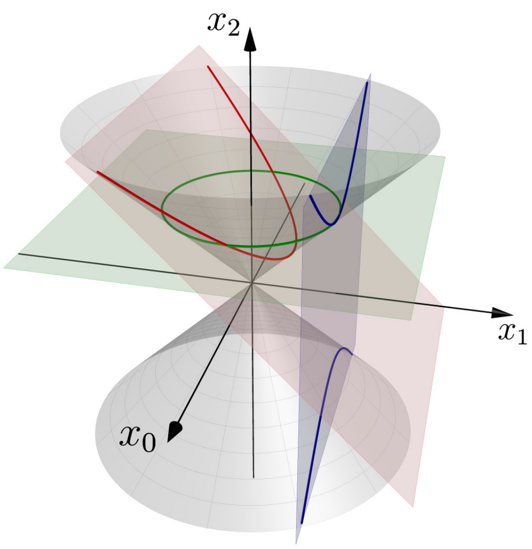

Fig. 1⋄5. Real projective conic.

1.3.1 Projective closure of affine variety.

The affine space $\mathbb{A}^n = \mathbb{A}(\Bbbk^n)$ with coordinates

$$(x_1, x_2, \dots, x_n)$$

can be considered as the standard affine chart U_0 in the projective space $\mathbb{P}_n = \mathbb{P}\left(\Bbbk^{n+1}\right)$ with homogeneous coordinates $(x_0 : x_1 : \dots : x_n)$. Every affine algebraic hypersurface $S = V(f) \subset \mathbb{A}^n$, where $f(x_1, x_2, \dots, x_n)$ is a (non-homogeneous) polynomial of degree d, admits the canonical extension to the projective hypersurface $\overline{S} = V(\overline{f}) \subset \mathbb{P}_n$ called the *projective closure* of S and defined by the homogeneous polynomial $\overline{f}(x_0, x_1, \dots, x_n) \in S^d V^*$ of the same degree d such that

$$\overline{f}(1, x_1, \dots, x_n) = f(x_1, x_2, \dots, x_n)\,.$$

This polynomial is constructed as follows: write f as

$$f(x_1, x_2, \dots, x_n) = f_0 + f_1(x_1, x_2, \dots, x_n) + f_2(x_1, x_2, \dots, x_n) + \cdots + f_d(x_1, x_2, \dots, x_n)$$

[1] Move x_1^2 to the right hand side of (1-2) and divide the both sides by $(x_2 + x_1)^2$.

where every component f_i is homogeneous of degree i, and put

$$\overline{f}(x_0, x_1, \dots, x_n) = f_0 \cdot x_0^d + f_1(x_1, x_2, \dots, x_n) \cdot x_0^{d-1} + \cdots + f_d(x_1, x_2, \dots, x_n).$$

Note that $\overline{S} \cap U_0 = S$ and the complement $\overline{S} \smallsetminus S = \overline{S} \cap U_0^{(\infty)}$ is cut out of $\overline{S}$ by the infinite hyperplane $x_0 = 0$ of the chart U_0. In terms of the standard homogeneous coordinates $(x_1 : x_2 : \cdots : x_n)$ on the infinite hyperplane, the intersection with $\overline{S}$ is described by the homogeneous equation

$$f_d(x_1, x_2, \dots, x_n) = 0$$

of degree d, that is, by the vanishing of top degree homogeneous component of the polynomial f describing S. Thus, the infinite points of $\overline{S}$ are nothing else than the *asymptotic directions* of affine hypersurface S.

For example, the projective closure of affine cubic curve $x_1 = x_2^3$ is the projective cubic $x_0^2 x_1 = x_2^3$. The latter has exactly one infinite point $p_\infty = (0 : 1 : 0)$. In the standard chart U_1, which covers this point, the curve looks like the semi-cubic parabola $x_0^2 = x_2^3$ with a cusp at p_∞.

1.3.2 Space of hypersurfaces. Since proportional polynomials define the same hypersurfaces $V(f) = V(\lambda f)$, the projective hypersurfaces of a fixed degree d can be viewed as the points of projective space $\mathcal{S}_d = \mathcal{S}_d(V) \overset{\text{def}}{=} \mathbb{P}(S^d V^*)$, which is called the *space of degree d hypersufaces* in $\mathbb{P}(V)$.

EXERCISE 1.7. Find $\dim \mathcal{S}_d(V)$ assuming that $\dim V = n + 1$.

Projective subspaces of $\mathcal{S}_d$ are called *linear systems* of hypersurfaces. For example, all degree d hypersurfaces passing through a given point $p \in \mathbb{P}(V)$ form a linear system of codimension one, i.e., a hyperplane in $\mathcal{S}_d$, because the equation $f(p) = 0$ is linear in $f \in S^d V^*$. Every hypersurface laying in a linear system spanned by $V(f_1)$, $V(f_2)$, … , $V(f_m)$, is given by equation of the form

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m = 0, \quad \text{where } \lambda_1, \lambda_2, \dots, \lambda_m \in \Bbbk.$$

In particular, any such a hypersurface contains the intersection locus $V(f_1) \cap V(f_2) \cap \dots \cap V(f_m)$. The points of this intersection are called the *base points* of the linear system. Traditionally, linear systems of dimensions 1, 2, 3 are called *pencils*, *nets*, and *webs* respectively.

EXERCISE 1.8. Show that each pencil of hypersurfaces contains a hypersurface passing through an arbitrarily prescribed point.

CAUTION 1.1. It should be kept in mind that if the ground field is not algebraically closed, then some polynomials of degree d may determine nothing geometrically reminiscent of a hypersurface of degree d. For example, the equation $x_0^2 + x_1^2 = 0$ over $\mathbb{R}$ describes the empty set $\varnothing$ on the projective line $\mathbb{P}_1$, and the one point set $(0 : 0 : 1)$ in the projective plane $\mathbb{P}_2$. Even over an algebraically closed field, some distinct points $f \neq g$ in $\mathbb{P}(S^d V^*)$ produce the same zero set $V(f) = V(g)$ in $\mathbb{P}(V)$. For example, the non-proportional polynomials $x_0^2 x_1$ and $x_0 x_1^2$ define the same two-point set $\{(0 : 1), (1 : 0)\}$ on $\mathbb{P}_1$. We postpone the discussion of geometric concepts avoiding such problems up to **??**.

1.3.3 Working example: unordered collections of points on the line. Let $U = \Bbbk^2$ with the standard coordinates x_0, x_1. Every set of d not necessary distinct points $p_1, p_2, \dots, p_d \in \mathbb{P}_1 = \mathbb{P}(U)$ is the zero set of homogeneous polynomial of degree d

$$f(x_0, x_1) = \prod_{\nu=1}^{d} \det(x, p_\nu) = \prod_{\nu=1}^{d} (p_{\nu,1} x_0 - p_{\nu,0} x_1), \quad \text{where } p_\nu = (p_{\nu,0} : p_{\nu,1}), \tag{1-3}$$

which is predicted by the set uniquely up to a scalar factor. We say that the points p_i are the *roots* of f. Each non-zero homogeneous polynomial of degree d has at most d distinct roots on $\mathbb{P}_1$. If the ground field $\Bbbk$ is algebraically closed, the number of roots[1] equals d, and sending a collection of points $p_1, p_2, \dots, p_d$ to the polynomial (1-3) establishes the bijection between the non-ordered d-typles of points on $\mathbb{P}_1$ and the points of projective space $\mathbb{P}(S^d U^*)$.

For an arbitrary field $\Bbbk$, those collections where all d points coincide form a curve

$$C_d \subset \mathbb{P}_d = \mathbb{P}(S^d U^*)$$

called the *Veronese curve*[2] of degree d. It coincides with the image of the *Veronese embedding*

$$v_d : \mathbb{P}_1^\times = \mathbb{P}(U^*) \hookrightarrow \mathbb{P}_d = \mathbb{P}\left(S^d U^*\right), \quad \varphi \mapsto \varphi^d, \tag{1-4}$$

that takes a linear form $\varphi \in U^*$, whose zero set consists of one point $p = \operatorname{Ann}\varphi \in \mathbb{P}_1 = \mathbb{P}(U)$, to the d th power $\varphi^d \in S^d(U^*)$, whose zero set is the d-tiple point p.

Now assume that $\operatorname{char}\Bbbk = 0$, write polynomials $\varphi \in U^*$ and $f \in S^d(U^*)$ in the form[3]

$$\varphi(x) = \alpha_0 x_0 + \alpha_1 x_1, \quad f(x) = \sum_\nu a_\nu \cdot \binom{d}{\nu} x_0^{d-\nu} x_1^\nu,$$

and use $\alpha = (\alpha_0 : \alpha_1)$ and $a = (a_0 : a_1 : \dots : a_d)$ as homogeneous coordinates in the spaces $\mathbb{P}_1^\times = \mathbb{P}(U^*)$ and $\mathbb{P}_d = \mathbb{P}(S^d U^*)$ respectively. Then we get the following parameterization of the Veronese curve by the points of $\mathbb{P}_1^\times$:

$$(\alpha_0 : \alpha_1) \mapsto (a_0 : a_1 : \dots : a_d) = \left(\alpha_0^d : \alpha_0^{d-1}\alpha_1 : \alpha_0^{d-2}\alpha_1^2 : \dots : \alpha_1^d\right). \tag{1-5}$$

It shows that C_d consists of all those $(a_0 : a_1 : \dots : a_d) \in \mathbb{P}_d$ that form a geometric progression, i.e., such that the rows of matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{d-2} & a_{d-1} \\ a_1 & a_2 & a_3 & \dots & a_{d-1} & a_d \end{pmatrix}$$

are proportional. The condition $\operatorname{rk} A = 1$ is equivalent to the system of homogeneous quadratic equations $a_i a_{j+1} = a_{i+1} a_j$ saying that all 2×2-minors of A vanish. Thus, $C_d \subset \mathbb{P}_d$ is an algebraic projective variety rationally parameterized by the points of projective line. The intersection of C_d with an arbitrary hyperplane in $\mathbb{P}_d$ given by linear equation $A_0 a_0 + A_1 a_1 + \dots + A_d a_d = 0$ consists of the Veronese-images of roots $(\alpha_0 : \alpha_1) \in \mathbb{P}_1$ of homogeneous polynomial $\sum_\nu A_\nu \cdot \alpha_0^{d-\nu}\alpha_1^\nu$ of degree d. Since it has at most d roots, any $d+1$ distinct points on the Veronese curve do not lie in a hyperplane. This implies that for $2 \leqslant m \leqslant d+1$, any m distinct points of C_d span a subspace of dimension $m-1$ and do not lie in a subspace of dimension $(m-2)$.

EXERCISE 1.9. Make sure that this fails when $\operatorname{char}\Bbbk$ is positive and divides d.

If $\Bbbk$ is algebraically closed, C_d intersects any hyperplane in precisely d points (some of which may coincide). By this reason we say that C_d has degree d.

---

[1] Counted with *multiplicities*, where the multiplicity of a root p is defined as the maximal integer k such that $\det^k(x, p)$ divides f in $\Bbbk[x_0, x_1]$.

[2] It has several other names: *rational normal curve*, *twisted rational curve of degree d* etc

[3] Note that for $\operatorname{char}\Bbbk > 0$, the binomial coefficients $\binom{d}{\nu}$ may vanish and can not be factored out the coefficients of f.

Example 1.4 (Veronese conic)
The Veronese conic $C_2 \subset \mathbb{P}_2$ consists of quadratic trinomials $a_0x_0^2+2a_1x_0x_1+a_2x_1^2$ that are perfect squares of linear forms. It is given by the equation $D/4 = -\det\left(\begin{smallmatrix} a_0 & a_1 \\ a_1 & a_2 \end{smallmatrix}\right) = a_1^2 - a_0a_2 = 0$ and comes with the rational parametrization $a_0 = \alpha_0^2$, $a_1 = \alpha_0\alpha_1$, $a_2 = \alpha_1^2$.

1.4 Complementary subspaces and projections. Projective subspaces $K = \mathbb{P}(U)$ and $L = \mathbb{P}(W)$ in $\mathbb{P}_n = \mathbb{P}(V)$ are called *complementary*, if $K \cap L = \varnothing$ and $\dim K + \dim L = n-1$. For example, any two non-intersecting lines in $\mathbb{P}_3$ are complementary. In terms of the linear algebra, the complementarity of K, L means that the vector subspaces $U, W \subset V$ have zero intersection $U \cap V = 0$ and

$$\dim U + \dim W = \dim K + 1 + \dim L + 1 = n + 1 = \dim V\,,$$

i.e., $V = U \oplus W$. In this case every vector $v \in V$ has a unique decomposition $v = u + w$, where $u \in U$, $w \in W$. In particular, $v \notin U \cup W$ if and only if the both components u, w are non zero. Geometrically, this means that every point $p \notin K \sqcup L$ lies on a unique line intersecting the both subspaces K, L.

Exercise 1.10. Make it sure.

For a pair of complementary subspaces $K, L \subset \mathbb{P}_n$, the projection $\pi_L^K : (\mathbb{P}_n \smallsetminus K) \to L$ from K onto L acts identically on L and sends every point $p \notin K \sqcup L$ to the unique point $b \in L$ such that the line (pb) intersects K. In homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$ such that $(x_0 : x_1 : \ldots : x_m)$ are the coordinates in K and $(x_{m+1} : x_{m+2} : \ldots : x_n)$ are the coordinates in L, the projection π_L^K just removes the first $m+1$ coordinates x_ν, $0 \leqslant \nu \leqslant m$.

Example 1.5 (projecting a conic to a line)
Let $C, L \subset \mathbb{P}_2$ be the conic and line given by equations[1] $x_0^2 + x_1^2 = x_2^2$ and $x_0 = 0$ respectively. Consider the projection $\pi_L^p : C \to L$ of C to L from $p = (1 : 0 : 1) \in C$ and extend it to p by sending p to $(0 : 1 : 0) \in L$, the intersection point of L with the tangent line to C at p. In the standard affine chart U_2 this looks as on fig. 1⋄6. Clearly, π_L^p provides a bijection between L and C. This bijection is *birational*: the homogeneous coordinates of the corresponding points

$$q = (q_0 : q_1 : q_2) \in C$$
$$t = (0 : t_1 : t_2) = \pi_L^p(q) \in L$$

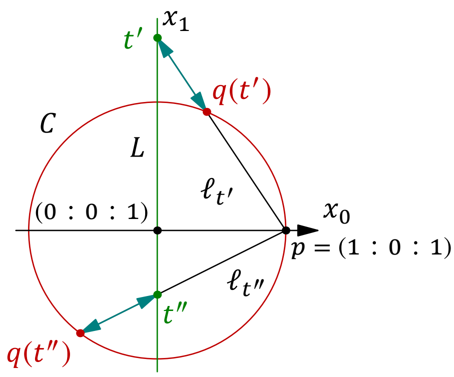

Fig. 1⋄6. Projecting a conic to a line.

are *rational* algebraic functions of each other:

$$(t_1 : t_2) = (q_1 : q_2 - q_0), \quad (q_0 : q_1 : q_2) = (t_1^2 - t_2^2 : 2t_1t_2 : t_1^2 + t_2^2)$$

Exercise 1.11. Check these formulas and use the second of them to list all integer solutions of the Pythagor equation $a^2 + a^2 = c^2$ up to common integer factor.

The invertible linear change of homogeneous coordinates by formulas

$$\begin{cases} a_0 = x_2 + x_0 \\ a_1 = x_1 \\ a_2 = x_2 - x_0 \end{cases} \qquad \begin{cases} x_0 = (a_0 - a_2)/2 \\ x_1 = a_1 \\ x_0 = (a_0 + a_2)/2 \end{cases}$$

[1] It is the same as in the Example 1.3 on p. 7 above.

transforms C to the Veronese conic $a_1^2 = a_0a_2$ from the Example 1.4 on p. 11 and turns the above parameterization to the standard parameterization of Veronese conic.

1.5 Linear projective transformations. Any linear isomorphism of vector spaces $F : U \xrightarrow{\sim} W$ produces well defined bijection $\overline{F} : \mathbb{P}(U) \xrightarrow{\sim} \mathbb{P}(W)$ called *a linear projective isomorphism*.

EXERCISE 1.12. Given two hyperplanes $L_1, L_2 \subset \mathbb{P}_n = \mathbb{P}(V)$ and a point $p \notin L_1 \cup L_2$, verify that a projection from p to L_2 induces a linear projective isomorphism $\gamma_p : L_1 \xrightarrow{\sim} L_2$.

THEOREM 1.1
For any two vector spaces U, W of the same dimension $n+1$ and two ordered collections of $n+2$ points $p_0, p_1, \dots, p_{n+1} \in \mathbb{P}(U)$, $q_0, q_1, \dots, q_{n+1} \in \mathbb{P}(W)$ such that no $n+1$ points of each collection lie in a hyperplane, there exists a unique up scalar factor linear isomorphism of vector spaces $F : U \xrightarrow{\sim} W$ such that $\overline{F}(p_i) = q_i$ for all i.

PROOF. Fix some vectors u_i, w_i representing the points p_i, q_i and chose the vectors $u_0, u_1, \dots, u_n$ and $w_0, w_1, \dots, w_n$ as the bases in U and W. The condition $\overline{F}(p_i) = q_i$ means that $F(u_i) = \lambda_i w_i$ for some non zero $\lambda_i \in \mathbb{k}$. Thus, the matrix of F in chosen bases is diagonal with $\lambda_0, \lambda_1, \dots, \lambda_n$ on the diagonal. Further, all coordinates x_i in the expansion $u_{n+1} = x_0u_0 + x_1u_1 + \dots + x_nu_n$ are non zero, because vanishing of x_k forces $n+1$ points p_j with $j \neq k$ lie in the hyperplane $x_k = 0$. The same holds for the expansion $w_{n+1} = y_0w_0 + y_1w_1 + \dots + y_nw_n$, certainly. The condition $F(u_{n+1}) = \lambda_{n+1}w_{n+1}$ implies that $\lambda_i x_i = \lambda_{n+1}y_i$ for all $0 \leqslant i \leqslant n$. Therefore the diagonal elements $\lambda_i = \lambda_{n+1} \cdot y_i / x_i$, $0 \leqslant i \leqslant n$, are uniquely determined by $\overline{F}$ up to non zero scalar factor λ_{n+1}. □

COROLLARY 1.1
Two linear isomorphisms of vector spaces $F, G : U \xrightarrow{\sim} W$ produce the same linear projective isomorphism $\overline{F} = \overline{G} : \mathbb{P}(U) \xrightarrow{\sim} \mathbb{P}(W)$ if and only if $F = \lambda G$ for some non zero $\lambda \in \mathbb{k}$. □

EXAMPLE 1.6 (AUTOMORPHISMS OF QUADRANGLE)
A figure formed by 4 points p_1, p_2, p_3, $p_4 \in \mathbb{P}_2$ any 3 of which are non-collinear and 6 lines joining the points like on fig. 1◇7 is called *a quadrangle*. The intersection points of its opposite sides:

$$q_1 = (p_1p_2) \cap (p_3p_4)$$
$$q_2 = (p_1p_3) \cap (p_2p_4)$$
$$q_3 = (p_1p_4) \cap (p_2p_3)$$

and 3 lines joining them form the *associated triangle* of the quadrangle. Every linear projective automorphism of $\mathbb{P}_2$ sending the quadrangle to itself permutes its vertexes, and every permutation of the vertexes is uniquely extended to a linear projective automorphism of $\mathbb{P}_2$ by the Theorem 1.1. Hence, the group of all linear projective automorphism of $\mathbb{P}_2$ sending the quadrangle to itself is naturally identified with the symmetric group S_4. Every transformation from this group permutes the vertexes of associated triangle. This leads to the surjective homomorphism of groups $S_4 \twoheadrightarrow S_3$. Its kernel is the

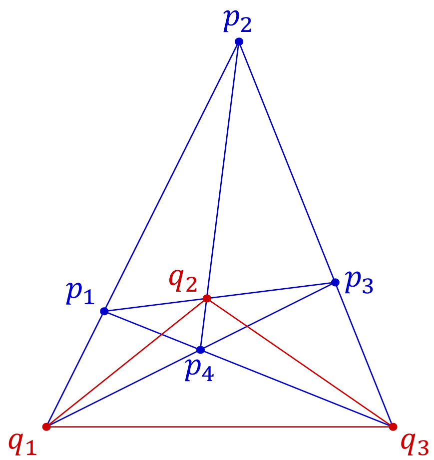

Fig. 1◇7. Quadrangle and associated triangle.

Klein's normal subgroup

$$V_4 = \{(1,2,3,4),\ (2,1,4,3),\ (3,4,1,2),\ (4,3,2,1)\} \lhd S_4$$

formed by the identity permutation and 3 pairs of independent transpositions. The transpositions (12), (13), (23) and 3-cycles (123), (132) from the group S_4 are mapped to the same transpositions (12), (13), (23) and 3-cycles (123), (132) from the group S_3, see fig. 1◇7.

1.5.1 Projective linear group. Linear projective automorphisms of $\mathbb{P}(V)$ form a group called the *projective linear group* of V and denoted $\mathrm{PGL}(V)$. It follows from the Theorem 1.1 that this group is isomorphic to the quotient of linear group $\mathrm{GL}(V)$ by the subgroup of scalar dilatations. A choice of basis in V identifies $\mathrm{GL}(V)$ with the group $\mathrm{GL}_{n+1}(\Bbbk)$ of non-degenerated square matrices. Then $\mathrm{PGL}(V)$ is identified with group $\mathrm{PGL}_{n+1}(\Bbbk)$ of the same matrices considered up to proportionality. Such a matrix A acts on a point $x = (x_0 : x_1 : \ \dots \ : x_n) \in \mathbb{P}_n$ via left multiplication of the coordinate column: $x \mapsto (Ax^t)^t = xA^t$, where M^t means the transposed M.

Example 1.7 (linear fractional transformations of line)
The group $\mathrm{PGL}_2(\Bbbk)$ consists of non-degenerated 2×2-matrices $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ with $\alpha\delta - \beta\gamma \neq 0$ considered up to a constant factor. Such a matrix acts on $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ by the rule

$$(x_0 : x_1) \mapsto (\alpha x_0 + \beta x_1 : \gamma x_0 + \delta x_1).$$

In the standard affine chart $U_1 \simeq \mathbb{A}^1$ this action performs the linear fractional transformation of the local coordinate $t = x_0/x_1$ by the rule $t \mapsto (\alpha t + \beta)/(\gamma t + \delta)$. Clearly, this transformation is not changed under rescaling of the matrix A. For any triple of distinct points q, r, s, there is a unique linear fractional map sending them to ∞, 0, 1 respectively. Indeed, this map is forced to take

$$t \mapsto \frac{t-r}{t-q} \cdot \frac{s-r}{s-q}. \tag{1-6}$$

1.5.2 Cross-ratio. Given two points $a = (a_0 : a_1)$, $b = (b_0 : b_1)$ on $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$, the difference of their affine coordinates in the standard chart U_1 is expressed trough the determinant of their homogeneous coordinates by the formula

$$a - b = \frac{a_0}{a_1} - \frac{b_0}{b_1} = \frac{a_0 b_1 - a_1 b_0}{a_1 b_1} = \frac{\det(a,b)}{a_1 b_1}.$$

For an ordered quadruple of distinct points $p_1, p_2, p_3, p_4 \in \mathbb{P}_1$, the quantity

$$[p_1, p_2, p_3, p_4] \overset{\text{def}}{=} \frac{(p_1 - p_3)(p_2 - p_4)}{(p_1 - p_4)(p_2 - p_3)} = \frac{\det(p_1, p_3) \cdot \det(p_2, p_4)}{\det(p_1, p_4) \cdot \det(p_2, p_3)} \tag{1-7}$$

is called the *cross-ratio* of the quadruple p_1, p_2, p_3, p_4. It follows from (1-6) that $[p_1, p_2, p_3, p_4]$ equals the affine coordinate of image of the point p_4 under the linear projective isomorphism sending p_1, p_2, p_3 to ∞, 0, 1 respectively. It can take any value except for ∞, 0, 1.

Exercise 1.13. Prove that two ordered quadruples of distinct points on $\mathbb{P}_1$ can be transformed one to the other by a linear projective automorphism if and only if they have equal cross-ratios.

Since an invertible linear change of homogeneous coordinates is nothing but a linear projective automorphism, the right hand side of (1-7) does not depend on the choice of coordinates on $\mathbb{P}_1$. This forces the middle part of (1-7) to depend neither on the choice of affine chart containing the points[1] nor on the choice of local affine coordinate within the chart. The symmetric group S_4 acts on every given quadruple of points by permutations. It is clear from (1-7) that the Klein subgroup $V_4 \subset S_4$ preserves the cross-ratio: $[p_1, p_2, p_3, p_4] = [p_2, p_1, p_4, p_3] = [p_3, p_4, p_1, p_2] = [p_4, p_3, p_2, p_1]$.

EXERCISE 1.14. Check that the values of cross-ratio appearing under the action of V_4-cosets of identity, transpositions (12), (13), (23), and 3-cycles (123), (132) are related as follows:

$$\begin{aligned}
&[p_1, p_2, p_3, p_4] = [p_2, p_1, p_4, p_3] = [p_3, p_4, p_1, p_2] = [p_4, p_3, p_2, p_1] = \vartheta\\
&[p_2, p_1, p_3, p_4] = [p_1, p_2, p_4, p_3] = [p_3, p_4, p_2, p_1] = [p_4, p_3, p_1, p_2] = 1/\vartheta\\
&[p_3, p_2, p_1, p_4] = [p_2, p_3, p_4, p_1] = [p_1, p_4, p_3, p_2] = [p_4, p_1, p_2, p_3] = \vartheta/(\vartheta-1)\\
&[p_1, p_3, p_2, p_4] = [p_3, p_1, p_4, p_2] = [p_2, p_4, p_1, p_3] = [p_4, p_2, p_3, p_1] = 1-\vartheta\\
&[p_2, p_3, p_1, p_4] = [p_3, p_2, p_4, p_1] = [p_1, p_4, p_2, p_3] = [p_4, p_1, p_3, p_2] = (\vartheta-1)/\vartheta\\
&[p_3, p_1, p_2, p_4] = [p_1, p_3, p_4, p_2] = [p_2, p_4, p_3, p_1] = [p_4, p_2, p_1, p_3] = 1/(1-\vartheta)\,.
\end{aligned} \tag{1-8}$$

These formulas show that there are three special values[2] $[p_1, p_2, p_3, p_4] = -1,\ 2,\ 1/2$ preserved, respectively, by the transpositions (12), (13), (23) and cyclically permuted by the 3-cycles. Similarly, there are two special values preserved by the 3-cycles and interchanged by the transpositions. They satisfy the equivalent quadratic equations[3] $\vartheta = (\vartheta-1)/\vartheta \Leftrightarrow \vartheta^2 - \vartheta + 1 = 0 \Leftrightarrow \vartheta = 1/(1-\vartheta)$.

The five just listed values of $[p_1, p_2, p_3, p_4]$ are called *special*. The quadruples of points with such cross-ratios are also called *special*. The permutations of points in a non-special quadruple lead to 6 distinct values of the cross-ratio. For a special quadruple we get either 3 or 2 distinct values.

1.5.3 Harmomic pairs of points. A special quadruple of points $a, b, c, d \in \mathbb{P}_1$ with $[a, b, c, d] = -1$ is called *harmonic*. Geometrically, this means that b is the middle point of $[c, d]$ in the affine chart with the infinity at a. Algebraically, the harmonicity means that the cross-ratio is changed neither by the transposition (12), nor by the transposition (34), and each of these two properties forces the quadruple to be harmonic. Since the order preserving exchange of a, b with c, d keeps the cross-ratio fixed, the harmonicity is a symmetric binary relation on the set of non-ordered pairs of distinct points in $\mathbb{P}_1$.

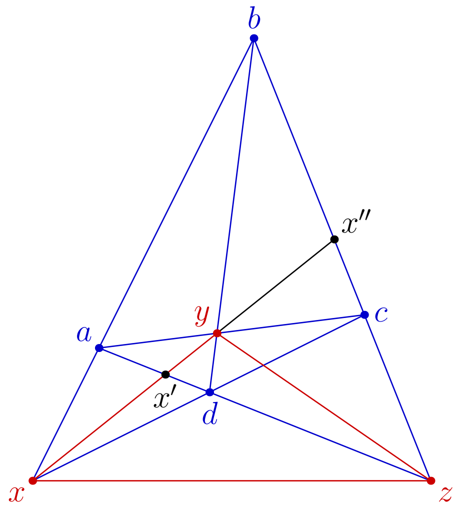

Fig. 1⋄8. Harmonic pairs of sides.

PROPOSITION 1.2 (HARMONICITY IN QUADRANGLE)
For any quadrangle a, b, c, d on $\mathbb{P}_2$ and its associated triangle x, y, z, the sides of quadrangle are harmonic to the sides of triangle in the pencils of lines passing through the vertexes of triangle.

PROOF. We verify the proposition at the vertex x. The pencil of lines passing through x is parameterized by the points of line (ad) by sending a point $p \in (ad)$ to the line (xp). We have to

[1] Algebraically, this means that all four values $p_1, p_2, p_3, p_4 \in \Bbbk$ are finite.
[2] They satisfy the equations $\vartheta = 1/\vartheta$, $\vartheta = \vartheta/(\vartheta-1)$, and $\vartheta = 1-\vartheta$.
[3] That is, coincide with two different from -1 cubic roots of one as soon those exist in $\Bbbk$.

check that $[a, d, z, x'] = -1$, see fig. 1◇8. Since the central projections from x and y preserve the cross-ratios, $[a, d, z, x'] = [b, c, z, x''] = [d, a, z, x']$. Since the transposition in the first pair of points does not change the cross-ratio, the latter equals -1. □

Comments to some exercises

Exrc. 1.4. The right hand side consists of $q^n + q^{n-1} + \cdots + q + 1$ points. The cardinality of the left hand side equals the number of non zero vectors in $\mathbb{F}_q^{n+1}$ divided by the number of non zero elements in $\mathbb{F}_q$, that is, $(q^{n+1}-1)/(q-1)$. We get the summation formula for geometric progression.

Exrc. 1.5. Every line passing through the origin of $\mathbb{R}^{n+1}$ intersects the unit semisphere $\sum x_i^2 = 1$, $x_0 \geqslant 0$. The lines laying in the hyperplane $x_0 = 0$ intersect the semisphere in two opposite points of the boundary. Any other line intersects the semisphere in exactly one internal point. Thus, $\mathbb{P}(\mathbb{R}^{n+1})$ can be obtained from the solid ball of dimension n by gluing together every pair of opposite points of its boundary sphere. In particular, the plane $\mathbb{P}_2 = \mathbb{P}(\mathbb{R}^3)$ is obtained from a square by gluing the opposite edges taken with opposite orientations, see fig. 5◇2.

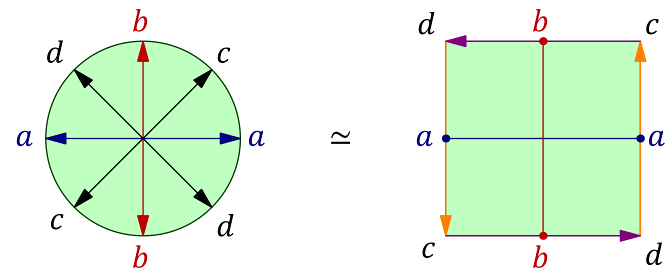

Fig. 5◇2. Gluing $\mathbb{P}(\mathbb{R}^3)$ from a square.

The same result is obtained by gluing a Möbius tape with a disk along the boundary circles, see fig. 5◇3.

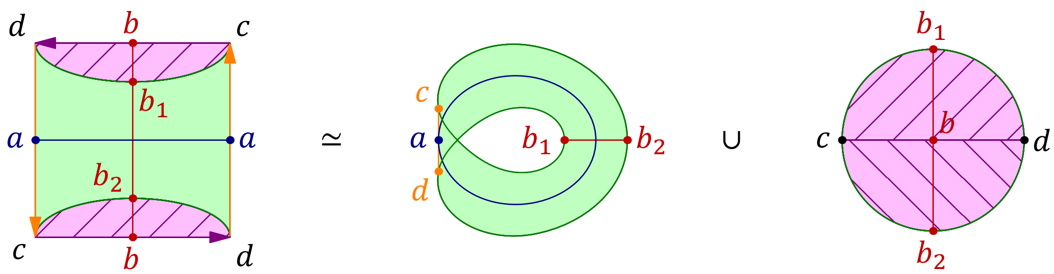

Fig. 5◇3. $\mathbb{P}(\mathbb{R}^3)$ as a Möbius tape glued to a disk along the boundary circle.

The solid ball of radius π in $\mathbb{R}^3$ is mapped onto the group SO_3 by sending a point P to the rotation about line (OP) by angle[1] $|OP|$ radians in the clockwise direction being viewed along $\overrightarrow{OP}$. This map is injective on internal points of the ball and identifies the opposite points of its boundary sphere.

Exrc. 1.6. Let $\mathbb{P}_n = \mathbb{P}(V)$, $K = \mathbb{P}(U)$, $L = \mathbb{P}(W)$ for some vector subspaces $U, W \subset V$. Then

$$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W) \geqslant \dim(K) + 1 + \dim(L) + 1 - n - 1 \geqslant 1\,.$$

Exrc. 1.7. $\binom{n+d}{d} - 1$.

Exrc. 1.8. In projective space any line does intersect any hyperplane, see the Exercise 1.6.

[1] We write $|AB|$ for the euclidean distance between the points A, B.

EXRC. 1.9. If char $\Bbbk = p > 0$ and $d = pm$, then $(\alpha_0 x_0 + \alpha_1 x_1)^d = (\alpha_0^p x_0^p + \alpha_1^p x_1^p)^m$ lies in the linear span of those monomials $x_0^\mu x_1^\nu$ whose exponents μ, ν both are divisible by p.

EXRC. 1.10. Let vector $v = u + w$ represent a point $p \in \mathbb{P}(V)$. Then $\ell = (u, w)$ passes through p and intersects K and L at u and w. Vice versa, if $v \in (a, b)$, where $a \in U$ and $b \in W$, then $v = \alpha a + \beta b$ and the uniqueness of the decomposition $v = u + w$ forces $\alpha a = u$ and $\beta b = w$. Hence $(ab) = \ell$.

EXRC. 1.12. Let $L_1 = \mathbb{P}(U)$, $L_2 = \mathbb{P}(W)$, $p = \mathbb{P}(\Bbbk \cdot e)$. Then $V = W \oplus \Bbbk \cdot e$, because of $p \notin L_2$. Projection from p is a projectivization of linear projection of V onto W along $\Bbbk \cdot e$. Since $p \notin L_1$, the restriction of this projection onto U has zero kernel. Thus, it produces linear projective isomorphism.

EXRC. 1.13. Let $[p_1, p_2, p_3, p_4] = [q_1, q_2, q_3, q_4]$. Write $\varphi_p, \varphi_q : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1$ for the linear projective automorphisms sending ∞, 0, 1 to the triples p_1, p_2, p_3 and q_1, q_2, q_3 respectively. Then $\varphi_p(p_4) = \varphi_q(q_4)$ and $\varphi_q^{-1} \circ \varphi_p$ sends p_1, p_2, p_3, p_4 to q_1, q_2, q_3, q_4. Vice versa, let a linear projective automorphism $\psi_{pq} : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1$ send q_1, q_2, q_3, q_4 to p_1, p_2, p_3, p_4. Write $\psi_p : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1$ for the linear projective automorphism sending p_1, p_2, p_3 to ∞, 0, 1. Then $\psi_p \circ \psi_{pq}$ takes

$$q_1, q_2, q_3, q_4 \mapsto \infty, 0, 1, [p_1, p_2, p_3, p_4].$$

Hence, $[p_1, p_2, p_3, p_4] = [q_1, q_2, q_3, q_4]$.

EXRC. 1.14. The map $(p_2, p_1, p_3) \mapsto (\infty, 0, 1)$ can be decomposed as the map $(p_1, p_2, p_3) \mapsto (\infty, 0, 1)$ followed by the map $(\infty, 0, 1) \mapsto (0, \infty, 1)$, which takes $\vartheta \mapsto 1/\vartheta$. Similarly, to permute (p_1, p_2, p_3) via the cycles (13), (23), (123), (132) we compose the map $(p_1, p_2, p_3) \mapsto (\infty, 0, 1)$ with the maps sending $(\infty, 0, 1)$ to $(1, 0, \infty)$, $(\infty, 1, 0)$, $(1, \infty, 0)$, $(0, 1, \infty)$ respectively, i.e., with the maps sending ϑ to $\vartheta/(\vartheta - 1)$, $1 - \vartheta$, $(\vartheta - 1)/\vartheta$, $1/(1 - \vartheta)$.