## Set 1. Tensors.

- **AG2**◇1. Let dim V = 3. Write  $S \subset \mathbb{P}_5 = \mathbb{P}(S^2V^*)$  for a variety of singular conics in  $\mathbb{P}_2 = \mathbb{P}(V)$ . Show that **a**) *S* is an algebraic hypersurface (and find the degree of *S*)
  - **b)** point  $C \in S$  is a smooth point of *S* iff the corresponding conic  $C \subset \mathbb{P}_n$  is a pair of crossing lines
  - c) tangent space  $T_c S \subset \mathbb{P}_5$  at a smooth point  $C \in S$  consists of all conics in  $\mathbb{P}_2$  passing through the singular point  $\ell_1 \cap \ell_2$  of  $C = \ell_1 \cup \ell_2$ .
- AG2 $\diamond$ 3 (spinor decomposition). Let  $V = \text{Hom}(U_{-}, U_{+})$ , where dim  $U_{\pm} = 2$ . Show that canonical direct sum decomposition of  $V \otimes V$  into symmetric and skew symmetric parts looks like

$$\underbrace{\left(\left(S^{2}U_{-}^{*}\otimes S^{2}U_{+}\right)\oplus\left(\Lambda^{2}U_{-}^{*}\otimes \Lambda^{2}U_{+}\right)\right)}_{S^{2}V}\bigoplus\underbrace{\left(\left(S^{2}U_{-}^{*}\otimes \Lambda^{2}U_{+}\right)\oplus\left(\Lambda^{2}U_{-}^{*}\otimes S^{2}U_{+}\right)\right)}_{\Lambda^{2}V}$$

AG2 $\diamond$ 4. For vector spaces *U*, *V* of finite dimensions construct canonical isomorphisms

 $\operatorname{Hom}(U \otimes \operatorname{Hom}(U, W), W) \simeq \operatorname{End}(\operatorname{Hom}(U, W)) \simeq \operatorname{Hom}(U, W \otimes \operatorname{Hom}(U, W)^*)$ 

and describe an element of End(Hom(U, W)) corresponding to a mapping  $U \otimes \text{Hom}(U, W) \to W$  that takes  $u \otimes \varphi \mapsto \varphi(u)$ .

AG2 \$ 5. For vector spaces U, V, W of finite dimensions construct canonical isomorphism

$$\operatorname{End}(U \otimes V \otimes W) \simeq \operatorname{Hom}(\operatorname{Hom}(U, V) \otimes \operatorname{Hom}(V, W), \operatorname{Hom}(U, W))$$

and describe a linear map  $\text{Hom}(U, V) \otimes \text{Hom}(V, W) \rightarrow \text{Hom}(U, W)$  corresponding to the identity endomorphism Id  $\in \text{End}(U \otimes V \otimes W)$ .

**AG26.** Let  $G = V(g) \subset \mathbb{P}_3 = \mathbb{P}(V)$  be a smooth quadric. Define a bilinear form  $\Lambda^2 \widetilde{g}$  on  $\Lambda^2 V$  by prescription

$$\Lambda^2 \widetilde{g}(v_1 \wedge v_2, w_1 \wedge w_2) \stackrel{\text{def}}{=} \det \begin{pmatrix} \widetilde{g}(v_1, w_1) & \widetilde{g}(v_1, w_2) \\ \widetilde{g}(v_2, w_1) & \widetilde{g}(v_2, w_2) \end{pmatrix} ,$$

a) Show that  $\Lambda^2 \tilde{g}$  is symmetric and non-degenerated.

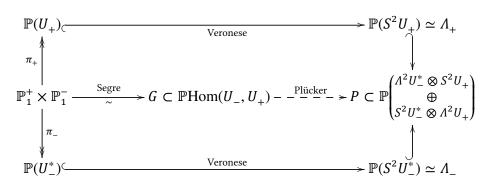
**b)** Write down an explicit Gram matrix of  $\Lambda^2 \tilde{g}$  in a standard monomial basis of  $\Lambda^2 \tilde{g}$  built from an orthonormal<sup>1</sup> basis of *V* 

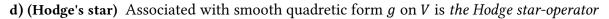
## Set 2. Plücker – Segre – Veronese interaction.

- **AG2**◆7. In the assumptions and notations of prb. AG2◆3 and prb. AG2◆6 take  $g(A) = \det A$  as the quadratic form on the space  $V = \operatorname{Hom}(U_{-}, U_{+})$ . Write  $\Lambda^2 g$  for the smooth quadratic form on  $\Lambda^2 V$  that sends  $v_1 \wedge v_2$  to the Gram determinant det  $\begin{pmatrix} \widetilde{g}(v_1, v_1) & \widetilde{g}(v_1, v_2) \\ \widetilde{g}(v_2, v_1) & \widetilde{g}(v_2, v_2) \end{pmatrix}$  and write  $P = \{\omega \in \Lambda^2 V \mid \omega \wedge \omega = 0\} \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$  for the Plücker quadric. Show that
  - a) the intersection of quadrics  $V(\Lambda^2 g) \cap P \subset \mathbb{P}_5$  consists of all lines in  $\mathbb{P}_3 = \mathbb{P}(V)$  tangent to the Segre quadric  $G = V(g) \subset \mathbb{P}_3$ .

<sup>&</sup>lt;sup>1</sup>that is, having the unit Gram matrix

- **b)** the Plücker embedding  $Gr(2, V) \cong P \subset \mathbb{P}(\Lambda^2 V)$  sends two line rulings of the Segre quadric *G* to a pair of distinct smooth conics  $C_{\pm} \subset P$  that are cut out of the Plücker quadric by a pair of complementary planes  $\Lambda_{-} = \mathbb{P}\left(S^2 U_{-}^* \otimes \Lambda^2 U_{+}\right)$  and  $\Lambda_{+} = \mathbb{P}\left(\Lambda^2 U_{-}^* \otimes S^2 U_{+}\right)$  embedded into  $\mathbb{P}(\Lambda^2 \operatorname{Hom}(U_{-}, U_{+}))$ via prb. AG2\$3
- c) both conics  $C_{-} \subset \mathbb{P}\left(S^{2}U_{-}^{*} \otimes \Lambda^{2}U_{+}\right)$  and  $C_{+} \subset \mathbb{P}\left(\Lambda^{2}U_{-}^{*} \otimes S^{2}U_{+}\right)$  are the images of the Veronese embeddings  $\mathbb{P}(U_{-}^{*}) \subset \mathbb{P}\left(S^{2}U_{-}^{*}\right)$  and  $\mathbb{P}(U_{+}) \subset \mathbb{P}\left(S^{2}U_{+}\right)$ , i.e. we have the following commutative diagram of the Plücker – Segre – Veronese interactions<sup>2</sup>:





$$*:\Lambda^2V\xrightarrow{\omega\mapsto\omega^*}\Lambda^2V,$$

defined by prescription  $\forall \omega_1, \omega_2 \in \Lambda^2 V \quad \omega_1 \wedge \omega_2^* = \Lambda^2 \tilde{g}(\omega_1, \omega_2) \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4$ , where  $e_1, e_2, e_3, e_4 \in V$  is an orthonormal basis for g. Verify that this definition does not depend on a choice of orthonormal basis, find eigenvalues and eigenspaces of \*, and show their place in the previous picture.

## Honorary problems

**AG28.** Generalise prb. AG2**1** onto the variety of singular quadrics in  $\mathbb{P}_n$  for any *n*. **AG29.** Prove the following Taylor expansion for the polynomial det(*A*) on the space of  $n \times n$ -matrices:

$$\det(\lambda A + \mu B) = \sum_{p+q=n} \lambda^p \mu^q \cdot \operatorname{tr} \left( \Lambda^p A \cdot \Lambda^q B^t \right) \,,$$

where  $\Lambda^{p}A$ ,  $\Lambda^{q}B$  are the matrices of operators induced by A, B on the spaces of homogeneous grassmannian polynomials of degrees p, q (matrix elements of  $\Lambda^p A$ ,  $\Lambda^q B$  are  $p \times p$  and  $q \times q$  minors of A, B numbered in such a way that complementary minors have equal numbers).

**AG210** (De Rahm's and Koszul's complexes). Choose a basis  $e_1, e_2, \ldots, e_n \in V$  and write  $x_i \in SV, \xi_i \in AV$ for the classes of  $e_i$  in symmetric and exterior algebras respectively. Let  $A = AV \otimes SV$ . Consider two linear mappings: the De Rahm differential  $d = \sum \xi_v \otimes \frac{\partial}{\partial x_v} : A \to A$  that takes  $\omega \otimes f \mapsto \sum_v \xi_v \wedge \omega \otimes \frac{\partial f}{\partial x_v}$ and the Koszul differential  $\partial = \sum \frac{\partial}{\partial \xi_v} \otimes x_v : A \to A$  that takes  $\omega \otimes f \mapsto \sum_v \frac{\partial \omega}{\partial \xi_v} \otimes x_v \cdot f$ .

- a) Show that *d* and  $\partial$  do not depend on a choice of basis and satisfy  $d^2 = 0$ ,  $\partial^2 = 0$ .
- **b)** Compute  $d\partial + \partial d$ .
- c) (Poincare lemma) Show that both homology spaces ker d/im d and ker  $\partial/\text{im } \partial$  are 1-dimensional, exhausted by the classes of constants  $\mathbb{k} \cdot 1 \otimes 1 \subset A$ .

<sup>&</sup>lt;sup>2</sup>Plücker is dashed, because it takes lines to points