Set 1. Projective spaces.

AG1⋄1. Let the ground field $\Bbbk$ consist of q elements. Find the total number of k-dimensional **a)** vector subspaces in $\Bbbk^n$ **b)** affine subspaces in $\mathbb{A}^n$ **c)** projective subspaces in $\mathbb{P}_n$. (Hint: to begin with, take $k = 0,\ 1,\ 2,\ \dots$)

AG1⋄2. Compute the limits of the previous answers as $q \to 1$.

AG1⋄3. Find a geometric condition on 3 lines ℓ_1, ℓ_2, ℓ_3 in $\mathbb{P}_2 = \mathbb{P}(V)$ necessary and sufficient for existence a coordinate system in V such that each ℓ_i becomes the infinite line for the standard chart $U_i = U_{x_i}$ in these coordinates.

AG1⋄4. Given a line ℓ and a point $p \notin \ell$, is it possible to draw the line parallel to ℓ and passing through p using only the ruler?

AG1⋄5. There are two points on a wall and a ruler whose length is significantly shorter than a distance between the points. Draw a straight line joining the points.

AG1⋄6. A point P and two non-parallel lines are drawn on a sheet of paper. The intersection point Q of the lines is far outside the sheet border. Using only the ruler, draw a part of line PQ laying inside the sheet.

AG1⋄7 (the Papus theorem). Let points a_1, b_1, c_1 be collinear and poins a_2, b_2, c_2 be collinear as well. Show that intersection points $\left(a_1b_2\right) \cap \left(a_2b_1\right)$, $\left(b_1c_2\right) \cap \left(b_2c_1\right)$, $\left(c_1a_2\right) \cap \left(c_2a_1\right)$ are collinear too.

AG1⋄8. Formulate and prove the dual statement[1] to the Papus theorem.

AG1⋄9 (1st theorem of Dezargus). Given 2 triangles $A_1B_1C_1$ and $A_2B_2C_2$ on $\mathbb{P}_2$, show that three intersection points $(A_1B_1) \cap (A_2B_2)$, $(B_1C_1) \cap (B_2C_2)$, $(C_1A_1) \cap (C_2A_2)$ are collinear iff three lines (A_1A_2), (B_1B_2), (C_1C_2) are intersecting at one point[2].

Hint. For $\Bbbk = \mathbb{R}$ simplify the configuration by moving 3 intersection points to infinity, then use the Euclidean geometry. For an arbitrary $\Bbbk$ investigate the fixed point set of the linear projective automorphism sending A_1, B_1, C_1 to A_2, B_2, C_2 and preserving the intersection point $(A_1A_2) \cap (B_1B_2) \cap (C_1C_2)$.

AG1⋄10 (2nd theorem of Dezargus). Let a line ℓ pass through three distinct points p, q, r but do not contain any of three other distinct points a, b, c. Show that lines (ap), (bq), (cr) are intersecting at one point iff there exists an involution of ℓ that exchanges p, q, r with intersection points of ℓ with lines (bc), (ca), (ab) respectively.

Set 2. Conics and quadrics.

AG1⋄11. Put real Euclidian plane $\mathbb{R}^2$ into $\mathbb{CP}_2$ as the real part of the standard chart $U_0 = \mathbb{C}^2$.

a) Find two points $A_\pm \in \mathbb{CP}_2$ laying on all conics visible in $\mathbb{R}^2$ as the circles.

b) Let a conic $C \subset \mathbb{CP}_2$ have at least 3 non-collinear points in $\mathbb{R}^2$ and pass through $A_\pm$. Show that $C \cap \mathbb{R}^2$ is a circle.

AG1⋄12. Given 5 lines without triple intersections on $\mathbb{P}_2$, how many conics do touch them all?

AG1⋄13. Consider a circle C in the real euclidean plane $\mathbb{R}^2$ and write D for a disc bounded by C. Using ruler and compasses, draw a polar line to a given point $p \in D$ and find a pole of a given line ℓ that does not intersect C. (All the polarities are w.r.t. C.)

AG1⋄14. Using only the ruler, draw a line passing through a given point p and touching a given conic C. Consider two cases: **a)** $p \notin C$ **b)** $p \in C$.

AG1⋄15. Line (pq) intersects conic C in points r, s. Assuming that all 4 points p, q, r, s are distinct, show that p lies on the polar line of q w.r.t. C iff $\{p, q\}$ are harmonic to $\{r, s\}$ (i.e. $[p, q; r, s] = -1$).

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[1] that holds in the dual space $\mathbb{P}_2^\times = \mathbb{P}(V^*)$ and dials with the annihilators of all the subspaces from the original statement

[2] pair of triangles with these properties is called *perspective*

AG1⋄16. Given 4 mutually skew[3] lines in 3D-space, how many lines does intersect them all? Consider the cases when 3D-space in question is: **a)** $\mathbb{CP}_3$ **b)** $\mathbb{RP}_3$ **c)** affine $\mathbb{C}^3$ **d)** affine $\mathbb{R}^3$. Find all possible answers and indicate those which are stable w.r.t. small perturbation of the 4 given lines.

AG1⋄17. How many solutions have equations **a)** $x_0^2+x_1^2+x_2^2+x_3^2=0$ **b)** $x_1^2+x_2^2+x_3^2=-1$ over the field $\mathbb{F}_9$, which consist of 9 elements $a+b\sqrt{-1}$, $a,b=-1,0,1$, added and multiplied modulo 3.

Honorary problems

AG1⋄18*. Using only the ruler, draw a triangle inscribed in a given smooth conic Q and with sides a, b, c passing through 3 given points A, B, C. How many solutions may have this problem?

Hint. Start `naive' drawing from any $p\in Q$ and denote by $\gamma(p)$ the return point after passing trough A,B,C. Is the mapping $p\mapsto\gamma(p)$ a homography?

AG1⋄19*. Formulate and solve projectively dual problem to the previous one.

AG1⋄20* (Rational normal curve). Verify that the following curves $C\subset\mathbb{P}_d$ can be moved isomorphically to each other by appropriate linear projective automorphisms of $\mathbb{P}_d$.

a) Write U for the space of linear forms $\alpha_0t_0+\alpha_1t_1$ in (t_0,t_1) and use $(\alpha_0:\alpha_1)$ as a homogeneous coordinate on $\mathbb{P}_1=\mathbb{P}(U)$. Also, consider the space S^dU, of homogeneous forms in (t_0,t_1) of degree d, write these forms as $\sum\limits_{n=0}^{d}\binom{d}{n}a_nt_0^nt_1^{d-n}$, where $\binom{n}{k}$ are the binomial coefficients, and use $(a_0:a_1:\ \dots\ :a_n)$ as homogeneous coordinates on $\mathbb{P}_d=\mathbb{P}(S^dU)$. Then C is the image of the Veronese map $c_d:\mathbb{P}(U)\to\mathbb{P}(S^dU)$, which takes $\psi\mapsto\psi^d$.

b) In the notations from (a), $C\subset\mathbb{P}(S^dU)$ is given by the condition $\operatorname{rk}\begin{pmatrix}a_0 & a_1 & a_2 & \dots & a_{d-1}\\ a_1 & a_2 & a_3 & \dots & a_d\end{pmatrix}=1$.

c) In the notations from (a), C is an image of any map $\mathbb{P}(U)\to\mathbb{P}(S^dU)$ given in homogeneous coordinates by a rule $t=(\alpha_0:\alpha_1)\longmapsto(f_0(\alpha):f_1(\alpha):\ \dots\ :f_d(\alpha))$, where $f_0,f_1,\dots,f_d$ is any collection of linearly independent homogeneous polynomials in $\alpha=(\alpha_0,\alpha_1)$ of degree d.

d) Pick up a collection of $(d+1)$ distinct points $p_0,p_1,\dots,p_d\in\mathbb{P}_1=\mathbb{P}(\mathbb{k}^2)$, $p_\nu=(\alpha_\nu:\beta_\nu)$. Then C is the image of mapping $\varphi_{p_0,p_1,\dots,p_d}:\mathbb{P}_1\to\mathbb{P}_d$ that takes

$$x=(x_0:x_1)\mapsto\big(1/\det(p_0,x):1/\det(p_1,x):\ \cdots\ :1/\det(p_d,x)\big)\,,$$

where $\det(p_\nu,x)\overset{\text{def}}{=}\alpha_\nu x_1-\beta_\nu x_0$.

e) Pick up any collection of $(d+3)$ distinct points $p_1,p_2,\dots,p_n$, a, b, $c\in\mathbb{P}_n$ such that no $(n+1)$ of them lie in a shared hyperplane and write $\ell_i\simeq\mathbb{P}_1$ for a pencil of hyperplanes passing through all points p_ν except for p_i. Points a, b, c provide the lines ℓ_ν with compatible homographies

$$\psi_{ij}:\ell_j\xrightarrow{\sim}\ell_i$$

sending 3 hyperplanes passing through a, b, c from the pencil ℓ_j to the similar 3 hyperplanes of ℓ_i. Then the curve C is drawn by the intersection point of d corresponding to each other hyperplanes of all the pencils: $C=\bigcup\limits_{H\in\ell_1}H\cap\psi_{21}(H)\cap\ \dots\ \cap\psi_{n1}(H)$.

[3]in projective space this means «non-intersecting», in affine space this means «not laying in a shared plane»