Curso de Inverno — ICMC/USP São Carlos

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SOME GEOMETRIC FACES OF ALGEBRA AND ALGEBRAIC FACES OF GEOMETRY

In Autumn 2012 Sasha Ananin asked me to give an intensive IUM-style course on basic projective geometry and attendant algebra at the São Carlos. I have made an attempt to combine a review of classical projective varieties: quadrics, Segre, Veronese, and Grassmannian with a self-contained introduction to linear, multilinear, and polynomial algebra and supplement this mixture with real affine convex geometry. The important part of this course consists of exercises and home task problems. The material of almost all exercises is intensively used along the course. Independent solution of problems followed by discussion of problems with teachers allows to understand the things deeper. The exciting title belongs to Carlos Henrique Grossi Ferreira. Without his support these lectures would be impossible.

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§1 Numbers, functions, spaces, and figures

1.1 Fields. An interaction between algebra and geometry stars with fixation of *the ground field* denoted by $\Bbbk$. This is a set of «constants» or «scalars» with two operations: addition $(+)$ and multiplication $(\cdot)$ that share the following properties of rational numbers:

$$\begin{array}{ccc} a+b=b+a & \text{(commutativity)} & a\cdot b=b\cdot a \\ (a+b)+c=a+(b+c) & \text{(associativity)} & (a\cdot b)\cdot c=a\cdot(b\cdot c) \\ \exists\, 0 : \forall\, a \quad 0+a=a & \text{(neutrals)} & \exists\, 1 : \forall\, a \quad 1\cdot a=a \\ \forall\, a\ \exists -a : a+(-a)=0 & \text{(opposites)} & \forall\, a\neq 0\ \exists\, a^{-1} : a\cdot a^{-1}=1 \end{array}$$

$$\text{distributivity: } a\cdot(b+c)=a\cdot b+a\cdot c$$

$$\text{non-triviality: } 0\neq 1$$

Thus, the elements of $\Bbbk$ can be added, subtracted, multiplied, and divided in the manner of rational numbers.

Most closed interaction between Algebra and Geometry[1] takes place when $\Bbbk=\mathbb{C}$ is the field of complex numbers or, more generally, when $\Bbbk$ is algebraically closed. However, a significant piece of classical geometry makes sense over any field.

Over a finite field $\Bbbk$, all spaces and figures become finite sets and geometric and/or algebraic theorems obtain combinatorial flavour. To make these lectures self-contained, let us remember some details concerning finite fields.

Example 1.1

Residue field $\mathbb{F}_7=\mathbb{Z}/(7)$ consist of 7 residues $[0]\,,\,[1]\,,\,[2]\,,\,\dots\,,\,[6]$ modulo 7. They ere added and subtracted like the hours rounded about clock face dial:

$$[2]-[5]=-[3]=[-3]=[4]\,.$$

The multiplication is less visual but it is still true that, say,

$$[2]+[2]+[2]+[2]=[2]\cdot[4]=[1]\,.$$

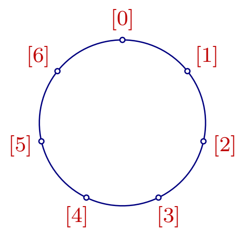

Fig. 1◇1.

Tables of squires and inverses are looking as follows

x	$[0]$	$[\pm1]$	$[\pm2]$	$[\pm3]$
x^2	$[0]$	$[1]$	$[4]$	$[2]$

x	$[-3]$	$[-2]$	$[-1]$	$[1]$	$[2]$	$[3]$
$1/x$	$[2]$	$[3]$	$[-1]$	$[1]$	$[-3]$	$[-2]$

Exercise 1.1. In $\mathbb{F}_5=\mathbb{Z}/(5)$ compute $2013\cdot[2]$ and $[2]^{2013}$ (i.e. 2013-tiple sum and product of $2\ (\mathrm{mod}\ 5)$ with itself).

1.1.1 Residues: integer numbers. The ring of residues $\mathbb{Z}/(m)$ does exist for any integer $m\geqslant 2$. It consists of equivalence classes $[k]$, $k\in\mathbb{Z}$, such that $[k]=[n]$ iff $k=n+m\cdot t$ for some $t\in\mathbb{Z}$. These classes are exhausted by $[0]\,,\ [1]\,,\,\dots\,,\,[m-1]$. The operations are well defined by the rules $[k]+[n]\overset{\text{def}}{=}[k+n]\,,\,[k]\cdot[n]\overset{\text{def}}{=}[kn]$.

Exercise 1.2. Verify that the results depend only on the classes but not on the particular choices of elements inside them.

[1] even an equivalence in some precise sense

Proposition 1.1
$\mathbb{Z}/(m)$ is a field iff m is *prime*[1].

Proof. Given $[a] \neq [0]$, to find $[x] = [a]^{-1}$, which satisfies $[a][x] = [1]$, means to solve an equation $ax + my = 1$ in $x, y \in \mathbb{Z}$. The minimal positive integer of the form $ax + my$, $x, y \in \mathbb{Z}$, is g.c.d.(a, m). Hence, $[a]$ is invertible iff g.c.d.$(a, m) = 1$. This holds for each $a = 1, 2, \dots, (m-1)$ iff m is prime. □

1.1.2 Residues: polynomials. Let $\Bbbk$ be any field and $\Bbbk[x]$ denote the ring of polynomials in x with coefficients from $\Bbbk$.

For any non-constant polynomial $f \in \Bbbk[x]$ one can form the residue ring $\Bbbk[x]/(f)$ in the same manner as above. The elements of $\Bbbk[x]/(f)$ are the equivalence classes $[g]$, $g \in \Bbbk[x]$, such that $[g] = [h]$ iff $g = h + f \cdot q$ for some $q \in \Bbbk[x]$. These classes are exhausted by $[g]$ with $\deg g < \deg f$. In other words, if $\deg f = n$, then all the classes

$$[\alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1}] = \alpha_0 + \alpha_1 [x] + \cdots + \alpha_{n-1} [x]^{n-1} \tag{1-1}$$

are different for different choices of constants $\alpha_0, \alpha_1, \dots, \alpha_{n-1} \in \Bbbk$ and exhaust the whole of $\Bbbk[x]/(f)$.

Proposition 1.2
$\Bbbk[x]/(f)$ is a field iff f is *irreducible*[2].

Proof. Given $[g] \neq [0]$, to find $[r] = [g]^{-1}$, which satisfies $[g][r] = [1]$, means to solve an equation $gr + fs = 1$ in $r, s \in \Bbbk[x]$. The monic[3] polynomial of smallest degree representable as $gr + fs$, $r, s \in \Bbbk[x]$, is g.c.d.(g, f). Hence, $[g]$ is invertible iff g.c.d.$(g, f) = 1$. This holds for each $g \neq 0$ with $\deg g < \deg f$ iff f is irreducible. □

1.1.3 Primitive field extensions. Since $f([x]) = [f(x)] = 0$ in $\Bbbk[x]/(f)$, we can treat expressions (1-1) as polynomials of degree $< \deg f$ in $[x]$ added and multiplied by the usual distribution rules modulo the relation $f([x]) = 0$, which allows to reduce the degree of an expression as soon it becomes $\geqslant \deg f$.

Example 1.2
Put $\mathbb{Q}[\sqrt{2}] \stackrel{\text{def}}{=} \mathbb{Q}[x]/(x^2 - 2)$. Here $[x]$ satisfies $[x]^2 - 2 = [x^2 - 2] = [0]$, that is $[x]^2 = 2$, and we write $\sqrt{2}$ for $[x]$. The field consists of all $\alpha + \beta\sqrt{2}$ with $\alpha, \beta \in \mathbb{Q}$. We have

$$\left(\alpha_1 + \beta_1\sqrt{2}\right) + \left(\alpha_2 + \beta_2\sqrt{2}\right) = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)\sqrt{2}$$
$$\left(\alpha_1 + \beta_1\sqrt{2}\right) \cdot \left(\alpha_2 + \beta_2\sqrt{2}\right) = (\alpha_1\alpha_2 + 2\beta_1\beta_2) + (\alpha_1\beta_2 + \alpha_2\beta_1)\sqrt{2}$$
$$\left(\alpha + \beta\sqrt{2}\right)^{-1} = \frac{\alpha}{\alpha^2 - 2\beta^2} - \frac{\beta}{\alpha^2 - 2\beta^2}\sqrt{2}\,.$$

Exercise 1.3. Show that: a) $x^2 - 2$ is irreducible in $\mathbb{Q}[x]$ b) $\alpha^2 - 2\beta^2 = 0$ only for $\alpha, \beta = 0$ (assuming $\alpha, \beta \in \mathbb{Q}$).

[1] or *irreducible*, that is $m = rs \Rightarrow r$ or s equals ± 1
[2] i.e. $f = gh \Rightarrow g$ or h is a constant
[3] we use this term for polynomials whose leading coefficient equals 1

Example 1.3
$\mathbb{R}[\sqrt{-1}\,] \overset{\text{def}}{=} \mathbb{R}[x]/(x^2+1)$ is the field of complex numbers $\mathbb{C}$. Indeed, $f = x^2+1$ is irreducible in $\mathbb{R}[x]$, because it has no real roots. Hence, $\mathbb{R}[x]/(f)$ is a field. Class $[x]$ satisfies $[x]^2 = -1$ and can be denoted by $\sqrt{-1}$. The field consists of $\alpha + \beta\sqrt{-1}$ with $\alpha, \beta \in \mathbb{R}$. The operations coincide with those of $\mathbb{C}$.

Example 1.4
We can repeat the previous for a finite residue field $\mathbb{F}_3 = \mathbb{Z}/(3)$ in the role of $\mathbb{R}$. Namely, $\mathbb{F}_3$ consist of 3 elements: $-1\,,0\,,1 \pmod 3$. Polynomial $f = x^2+1$ has no roots in $\mathbb{F}_3$, because $f(0) = 1$ and $f(\pm 1) = -1$. Thus, f is irreducible in $\mathbb{F}_3[x]$ and $\mathbb{F}_3[\sqrt{-1}\,] = \mathbb{F}_3[x]/(x^2+1)$ is a field. It consists of 9 elements $\alpha + \beta\sqrt{-1}$, where $\alpha, \beta = 0, 1, -1$, and is denoted by $\mathbb{F}_9$. The multiplication goes like in $\mathbb{C}$ but modulo 3. For example: $\left(1+\sqrt{-1}\right)^2 = -\sqrt{-1}\,, \quad \left(1+\sqrt{-1}\right)^{-1} = -1+\sqrt{-1}\,.$

Exercise 1.4. In $\mathbb{F}_9$ compute $\quad 2013\cdot\left(1+\sqrt{-1}\right)$ and $\left(1+\sqrt{-1}\right)^{2013}$.

1.1.4 Prime subfield and characteristic. Given a field $\mathbb{F}$, the intersection of all subfields $\Bbbk \subset \mathbb{F}$ is called *the prime subfield* of $\mathbb{F}$. It is the smallest subfield in $\mathbb{F}$ w.r.t. inclusions. The prime subfield contains all sums

$$\underbrace{1+1+\,\cdots\,+1}_{p}\,, \quad p \in \mathbb{N}\,. \tag{1-2}$$

If all these sums are different, then $\mathbb{F} \supset \mathbb{Z}$. Hence, $\mathbb{F} \supset \mathbb{Q}$ and the prime subfield of $\mathbb{F}$ equals $\mathbb{Q}$. In this case we say that $\mathbb{F}$ has *zero*[1] *characteristic* and write $\operatorname{char}\Bbbk = 0$.

If some of sums (1-2) coincide, then *the characteristic* $\operatorname{char}\mathbb{F}$ is defined as the smallest $p \in \mathbb{N}$ for which sum (1-2) vanishes. In this case we say that $\mathbb{F}$ has *finite* characteristic. The characteristic has to be a prime number because of the identity

$$\underbrace{1+1+\,\cdots\,+1}_{kn} = (\underbrace{1+1+\,\cdots\,+1}_{k})\cdot(\underbrace{1+1+\,\cdots\,+1}_{n})$$

(vanishing of L.H.S. implies vanishing of some factor in R.H.S.). Thus, the prime subfield of a field of characteristic p equals $\mathbb{F}_p = \mathbb{Z}/(p)$.

1.1.5 Frobenius homomorphism. Let $\operatorname{char}\Bbbk = p$ be finite. The *Frobenius map*[2]

$$F_p : \Bbbk \xrightarrow{\alpha\mapsto\alpha^p} \Bbbk \tag{1-3}$$

respects multiplication: $F_p(\alpha\beta) = (\alpha\beta)^p = \alpha^p\beta^p = F_p(\alpha)F_p(\beta)$ as well as summation:

$$F_p(\alpha+\beta) = (\alpha+\beta)^p = \alpha^p + \binom{p}{1}\alpha^{p-1}\beta + \,\cdots\, + \binom{p}{p-1}\alpha\beta^{p-1} + \beta^p = \alpha^p + \beta^p = F_p(\alpha) + F_p(\beta)$$

(since each $\binom{p}{k} = p(p-1)\cdots(p-k+1)/k!\,,\ 1 \leqslant k \leqslant p-1$, is divisible by p).

[1]or infinite (in some contexts)
[2]or just *Frobenius* for short

Frobenius keeps fixed each element $[n] = n \cdot 1$, $n \in \mathbb{N}$, of the prime subfield $\mathbb{F}_p = \mathbb{Z}/(p) \subset \Bbbk$, because $[n]^p = (\underbrace{[1] + [1] + \cdots + [1]}_{n})^p = \underbrace{[1]^p + [1]^p + \cdots + [1]^p}_{n} = \underbrace{[1] + [1] + \cdots + [1]}_{n} = [n]$ (this is known as *Fermat's little theorem*).

Exercise 1.5. Look at the Frobenius action on the field $\mathbb{F}_9$ from example 1.4. Does it coincide with «conjugation» $a + b\sqrt{-1} \mapsto a - b\sqrt{-1}$?

1.2 Vector spaces. A *vector space* over a field $\Bbbk$ is an abelian group V (with operation $+$) equipped with multiplication by elements $\lambda \in \Bbbk$ in such a way that

$$(\lambda + \mu)v = \lambda v + \mu v \qquad \lambda(v + w) = \lambda v + \lambda w$$
$$\lambda(\mu v) = (\lambda\mu)v \qquad 1 \cdot v = v\,.$$

We say that vectors $w_1, w_2, \dots, w_m$ *span* V, if each $v \in V$ can be expressed as a *linear combination* of w_i's, that is

$$v = \lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_m w_m \quad \text{for some } \lambda_1, \lambda_2, \dots, \lambda_m \in \Bbbk\,.$$

Vectors $u_1, u_2, \dots, u_n \in V$ are called *linearly independent*, if

$$\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_m u_m = 0 \;\Rightarrow\; \text{each } \lambda_i = 0\,.$$

Lemma 1.1 (exchange lemma)
Let $w_1, w_2, \dots, w_m$ span V and $u_1, u_2, \dots, u_n \in V$ be linearly independent. Then $m \geqslant n$ and after appropriate renumbering of w_i's vectors $u_1, u_2, \dots, u_n$, w_{n+1}, w_{n+2}, $\dots$, w_m do span V as well.

Proof. Assume inductively that for some $k < n$ the vectors $u_1, u_2, \dots, u_k$, w_{k+1}, w_{k+2}, $\dots$, w_m span V (the case $k = 0$ corresponds to the initial situation). Then u_{k+1} is a linear combination of these vectors. Since u_i's are linearly independent, this linear combination contains some w_j with non-zero coefficient. Renumbering w_ν's in order to have $j = k + 1$, we conclude that $k < m$ and w_{k+1} is a linear combination of vectors $u_1, u_2, \dots, u_{k+1}$, w_{k+2}, w_{k+3}, $\dots$, w_m. Hence, they span V and the inductive assumption holds for $k + 1$ as well. □

1.2.1 Bases and dimension. Linearly independent collection $v_1, v_2, \dots, v_n \in V$ that spans V is called a *basis* of V. Any collection of vectors spanning V clearly does contain some basis. By lemma 1.1 each linearly independent collection of vectors can be completed to some basis and all the bases consist of the same number of vectors. This number is called *dimension* of V and denoted $\dim V$.

Exercise 1.6. Find dimensions of a) the space of symmetric $n \times n$-matrices b) the space of skew-symmetric $n \times n$-matrices c) the space of homogeneous polynomials of degree d in n variables.

If $\dim V = n$, then it follows from lemma 1.1 that any n linearly independent vectors as well as any n vectors that span V form a basis for V.

1.2.2 Coordinates. Vectors $e_1, e_2, \dots, e_n \in V$ form a basis iff each vector $v \in V$ admits a unique expression $v = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$ with $x_i \in \Bbbk$. In this case the mapping

$$V \ni v = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n \;\mapsto\; (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Bbbk^n$$

provides an isomorphism between V and the *coordinate space* $\Bbbk^n$. Numbers $(\lambda_1, \lambda_2, \dots, \lambda_n)$ are called the *coordinates* of v w.r.t. the basis $\{e_i\}$.

Example 1.5 (Lagrange's interpolation)
Let $V = \{f \in \Bbbk[x] \mid \deg f \leqslant n\}$. Given $n+1$ distinct points $\alpha_0, \alpha_1, \dots, \alpha_n \in \Bbbk$, the polynomials

$$L_i(x) = \prod_{\nu \neq i}(x - \alpha_\nu) \Big/ \prod_{\mu \neq i}(\alpha_i - \alpha_\mu)\,, \quad i = 0,\, 1,\, \dots\,,\, n\,,$$

satisfy the relations

$$L_i(\alpha_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

which force the coefficients of any linear combination $g(x) = \sum \lambda_i L_i(x)$ to be equal to the values of g: $\lambda_i = g(\alpha_i)$. We conclude that L_i are linearly independent, thus, form a basis of V, thus, each $g \in V$ *does have* the expansion $g(x) = \sum g(\alpha_i) \cdot L_i(x)$.

1.2.3 Duality. A *covector* on a vector space V is *a linear map* $\varphi : V \to \Bbbk$, i.e. such that

$$\varphi(\lambda v + \mu w) = \lambda\varphi(v) + \mu\varphi(w) \quad \forall\, v, w \in V\,,\ \forall\, \lambda, \mu \in \Bbbk\,.$$

Covectors form a vector space called the *dual space* to V and denoted by V^*.

To emphasize the symmetric roles of V and V^* we will often write $\langle\, \varphi\,,\, v\,\rangle$ for the value $\varphi(v)$ and call it a *contraction* of a covector $\varphi \in V^*$ and a vector $v \in V$.

If $e_1, e_2, \dots, e_n \in V$ form a basis, then i-th *coordinate mapping* $x_i : V \to \Bbbk$, which takes $v = \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n$ to $x_i(v) \overset{\text{def}}{=} \lambda_i$, is a covector. This definition implies the following relations similar to those from example 1.5:

$$\langle\, x_i\,,\, e_j\,\rangle = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j\,. \end{cases} \tag{1-4}$$

Again, each covector of the form $\psi = \psi_1 x_1 + \psi_2 x_2 + \dots + \psi_n x_n \in V^*$ is forced to take $\psi(e_i) = \psi_i$. Hence x_i are linearly independent. Moreover, each $\psi \in V^*$ equals $\sum \psi(e_i) \cdot x_i$, because the both linear maps $V \to \Bbbk$ coincide on the basis $\{e_i\} \subset V$.

Exercise 1.7. Verify that if two linear maps $f, g : U \to W$ coincide on some collection of vectors spanning U, then they coincide everywhere on U.

Definition 1.1
Bases $x_1, x_2, \dots, x_n \in V^*$ and $e_1, e_2, \dots, e_n \in V$ satisfying (1-4) are called *dual* to each other.

Exercise 1.8. Given two collections of vectors $x_1, x_2, \dots, x_m \in V^*$ and $e_1, e_2, \dots, e_m \in V$ satisfying (1-4) show that they form dual bases if one of the following conditions holds:
a) $\{e_i\}$ span V b) $\{x_i\}$ span V^* c) $m = \dim V$ d) $m = \dim V^*$.

Proposition 1.3
As soon as $\dim V < \infty$ there is canonical isomorphism $V \xrightarrow{\sim} V^{**}$ sending $v \in V$ to the evaluation map $\mathrm{ev}_v : V^* \to \Bbbk$, which takes $\psi \mapsto \psi(v)$.

Proof. It sends any basis $\{e_i\}$ of V to a basis of V^{**} dual to the basis $\{x_i\}$ of V^* dual to $\{e_i\}$. □

Exercise 1.9. Given a subspace U in V or in V^* write $\operatorname{Ann} U$ for a subspace in the dual space (V^* or V respectively) defined as $\operatorname{Ann} U = \{\xi \mid \forall u \in U\ \langle\, \xi\,,\, u\,\rangle = 0\}$. Verify that the correspondence $U \mapsto \operatorname{Ann} U$ establishes a self-inverse bijection between the subspaces of dual spaces V and V^* that reverses the inclusions. In other words, show that $\operatorname{Ann}\operatorname{Ann} U = U$ and $U \subset W \iff \operatorname{Ann} U \supset \operatorname{Ann} W$. Moreover, show that it takes the linear spans of collections of subspaces to the intersections of their annihilators and vice versa.

1.3 Polynomials. Let $x_1, x_2, \dots, x_n \in V^*$ form a basis of V^*. By *a polynomial* on V we understand an element of the polynomial algebra

$$SV^* \stackrel{\text{def}}{=} \Bbbk[x_1, x_2, \dots, x_n]\,,$$

with coefficients in $\Bbbk$. Another choice of a basis in V^* leads to isomorphic algebra obtained from the initial one by an invertible linear change of variables.

We write $S^d V^* \subset SV^*$ for the subspace of homogeneous polynomials of degree d. Clearly, it is stable under linear changes of variables and has a basis $x^m \stackrel{\text{def}}{=} x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$ numbered by all collections $m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n_{\geqslant 0}$ of non-negative integers with $\sum m_i = d$.

Exercise 1.10. Show that $\dim S^d V^* = \binom{n+d-1}{d}$ as soon as $\dim V = n$.

In fact, the *symmetric powers* $S^d V^*$ and the whole *symmetric algebra* SV^* of the space V^* admit an intrinsic functorial coordinate-free definition but we postpone it until n° 3.3.1 on p. 58. Note that

$$SV^* = \bigoplus_{d \geqslant 0} S^d V^*\,, \quad \text{where } S^k V^* \cdot S^m V^* \subset S^{k+m} V^*\,.$$

1.3.1 Polynomial functions. Each polynomial $f = \sum\limits_m a_m x_1^{m_1} \dots x_n^{m_n} \in SV^*$ produces *a polynomial function* $V \to \Bbbk$ that takes

$$v \mapsto \sum_m a_m \left\langle\, x_1\,,\, v \,\right\rangle^{m_1} \dots \left\langle\, x_n\,,\, v \,\right\rangle^{m_n} \tag{1-5}$$

(evaluation of f at the coordinates of v). We get a homomorphism

$$SV^* \to \{\text{functions } V \to \Bbbk\}\,. \tag{1-6}$$

that takes a polynomial f to the function (1-5), which we will denote by the same letter f in spite of the next claim saying that this notation is not correct for finite fields.

Proposition 1.4
Homomorphism (1-6) is injective if an only if the ground field $\Bbbk$ is infinite.

Proof. If $\Bbbk$ consists of q elements, then the space of all functions $V \to \Bbbk$ consists of q^{q^n} elements whereas the polynomial algebra $\Bbbk[x_1, x_2, \dots, x_n]$ is definitely infinite. Hence, homomorphism (1-6) can not be injective.

Let $\Bbbk$ be infinite. For $n = 1$ each non zero polynomial $f \in \Bbbk[x_1]$ vanishes in at most $\deg f$ pints of $V \simeq \Bbbk$. Hence, the polynomial function $f : V \to \Bbbk$ is not the zero function. For $n > 1$ we proceed inductively. Write a polynomial $f \in \Bbbk[x_1, x_2, \dots, x_n]$ as a polynomial in x_n with the coefficients in $\Bbbk[x_1, x_2, \dots, x_{n-1}]$: $f = f(x_1, x_2, \dots, x_{n-1};\, x_n) = \sum\limits_\nu f_\nu(x_1, x_2, \dots, x_{n-1}) \cdot x_n^\nu$. Let the polynomial function $f : V \to \Bbbk$ vanish identically on $\Bbbk^n$. Evaluating the coefficients f_ν at any $w \in \Bbbk^{n-1}$, we get polynomial $f(w; x_n) \in \Bbbk[x_n]$ that produces identically zero function of

x_n. Hence, $f(w; x_n) = 0$ in $\Bbbk[x_n]$. Thus, all coefficients $f_\nu(w)$ are identically zero functions of $w \in \Bbbk^{n-1}$. By induction, they are zero polynomials. $\square$

Exercise 1.11. Give an explicit example of non-zero polynomial $f \in \mathbb{F}_p[x]$ that produces identically zero function $F_f : \mathbb{F}_p \to \mathbb{F}_p$.

1.3.2 Digression: list of finite fields. Each finite field $\Bbbk$ of characteristic p is a finite dimensional vector space over the prime subfield $\mathbb{F}_p \subset \Bbbk$. Let $\dim_{\mathbb{F}_p} \Bbbk = n$. Then $\Bbbk$ consists of p^n elements. Thus, cardinalities of finite fields are exhausted by the powers of primes.

For each power $q = p^n$ a field $\mathbb{F}_q$ of cardinality q is constructed as follows. Consider polynomial $f(x) = x^q - x \in \mathbb{F}_p[x]$ and use the construction n° 1.1.3 on p. 5 to build a field $\Bbbk \supset \mathbb{F}_p$ such that in $\Bbbk[x]$ f becomes a product of linear factors.

Exercise 1.12. Explain precisely how to built such a field $\Bbbk$.

Since $f'(x) = -1$, polynomial f has precisely q distinct roots in $\Bbbk$. They form a field, because for any two roots $\alpha = \alpha^q$ and $\beta = \beta^q$ we have

$$\alpha + \beta = \alpha^{p^n} + \beta^{p^n} = F_p^n(\alpha) + F_p^n(\beta) = F_p^n(\alpha + \beta) = (\alpha + \beta)^q ,$$
$$\alpha\beta = \alpha^q\beta^q = (\alpha\beta)^q , \quad (-\alpha) = (-\alpha)^q , \quad 1/\alpha = (1/\alpha)^q .$$

Using elementary group-theoretical arguments, we can say more:

Proposition 1.5
Any field $\mathbb{F}$ of cardinality $q = p^n$ is isomorphic to the field $\mathbb{F}_q$ constructed above.

Proof. Since the multiplicative group $\mathbb{F}^* \overset{\text{def}}{=} \mathbb{F} \smallsetminus \{0\}$ has order $q - 1$, each non-zero element $a \in \mathbb{F}^*$ satisfies an equation $a^{q-1} = 1$. Thus, $\mathbb{F}$ consists of q distinct roots of polynomial $x^q - x$.

Exercise 1.13 (group exponent). Let A be an abelian group and $a, b \in A$ have finite orders α and β respectively. Construct an element $c \in A$ of order l.c.m.(α, β). Deduce from this that if there exist the least common multiple[1] μ for the orders of all elements $a \in A$ (e.g. if A is finite) then there is an element $m \in A$ of order μ.

Write d for the *exponent*[2] of $\mathbb{F}^*$ and fix an element $\zeta \in \mathbb{F}^*$ of order d. We claim that $d = q - 1$: otherwise q elements of $\mathbb{F}$ would be the roots of polynomial $x^{d+1} - x$ of degree $d + 1 < q$. Thus, $\mathbb{F} = \{0, 1, \zeta, \zeta^2, \dots, \zeta^{q-2}\}$. Let $g \in \mathbb{F}_p[x]$ be the *minimal polynomial*[3] of ζ over $\mathbb{F}_p$. Then g is an irreducible factor of f in $\mathbb{F}_p[x]$. The evaluation map $\mathrm{ev}_\zeta : \mathbb{F}_p[x]/(g) \to \mathbb{F}$, $[h(x)] \mapsto h(\zeta)$, is well defined, because $g(\zeta) = 0$, and surjective, because $x^k \mapsto \zeta^k$. Thus, $\mathbb{F} \simeq \mathbb{F}_p[x]/(g)$.

Exercise 1.14. Verify that each non-zero homomorphism[4] of fields is injective.

On the other hand, since f has q roots in $\mathbb{F}_q$, substituting them into factorization $f = gr$, we conclude that g also has a root ξ in $\mathbb{F}_q$. Then the evaluation map $\mathrm{ev}_\xi : \mathbb{F}_p[x]/(g) \to \mathbb{F}_q$, $[h(x)] \mapsto h(\xi)$, is well defined injection. By the cardinality reasons $\mathbb{F}_p[x]/(g) \simeq \mathbb{F}_q$. $\square$

[1] it is called *an exponent* of group A
[2] see exrs. 1.13 above
[3] i.e. monic polynomial of minimal possible degree such that $g(\zeta) = 0$
[4] i.e. taking at least one non-zero value

1.4 Affine spaces. To pass to geometry we need a space that consists of *points* and allows to draw figures there. Associated with a vector space V of dimension n is an *affine space* $\mathbb{A}^n = \mathbb{A}(V)$ also called *an affinization* of V. By the definition, the points of $\mathbb{A}(V)$ are the vectors of V. The point corresponding to the zero vector is called *the origin* and denoted by O. All other points can be imagined as «the ends» of non zero vectors «drawn» from the origin.

Each polynomial $f \in SV^*$ on V produces the polynomial function $f : \mathbb{A}(V) \to \Bbbk$. The set of its zeros is denoted by $V(f) \stackrel{\text{def}}{=} \{p \in \mathbb{A}(V) \mid f(p) = 0\}$ and is called an *affine algebraic hypersurface*. An intersection of (any set[1] of) such hypersurfaces is called an *affine algebraic variety*. In other words, an algebraic variety is a figure $X \subset \mathbb{A}^n$ defined by an arbitrary system of polynomial equations.

The simplest hypersurface is an *affine hyperplane* given by affine linear equation $\langle \varphi , v \rangle = c$, where $\varphi \in V^*$ is non-zero covector and $c \in \Bbbk$. Such a hyperplane passes through the origin iff $c = 0$. In this case it coincides with the affine space $\mathbb{A}(\operatorname{Ann}\varphi)$ associated with the vector subspace $\operatorname{Ann}(\varphi) = \{v \in V \mid \langle \varphi , v \rangle = 0\}$ (comp. with exrs. 1.8 on p. 8). In general case, affine hyperplane $\varphi(v) = c$ is a shift of $\mathbb{A}(\operatorname{Ann}\varphi)$ by any vector u such that $\langle \varphi , u \rangle = c$.

Intersections of affine hyperplanes are called *affine subspaces* of $\mathbb{A}(V)$. Such a subspace is given by a system of affine linear equations $\left\langle \varphi_i , v \right\rangle = c_i$ and is either the empty set $\varnothing$ or a shift of $\mathbb{A}(\operatorname{Ann}\Phi)$, where $\Phi \subset V^*$ is a linear span[2] of all φ_i's, by any vector v satisfying all the equations $\left\langle \varphi_i , v \right\rangle = c_i$. Vice versa, any vector subspace $U \subset V$ produces an affine subspace $\mathbb{A}(U) \subset \mathbb{A}(V)$ given by a system of homogeneous linear equations $\langle \varphi , v \rangle = 0$, where φ runs through[3] $\operatorname{Ann} U \subset V^*$, and a family of its parallel shifts $w + \mathbb{A}(U)$, given by affine non-homogeneous equations $\langle \varphi , v \rangle = \langle \varphi , w \rangle$ (w is fixed, φ runs through $\operatorname{Ann} U$). Shifted subspaces $w + \mathbb{A}(U)$ and $u + \mathbb{A}(U)$ are either non-intersecting or coinciding. The latter means that $u - w \in U$, i.e. $u \equiv w \pmod U$. Thus, affine subspaces $w + \mathbb{A}(U)$, which are parallel to a given vector subspace $U \subset V$, stay in 1 – 1 correspondence with the vectors of the factor space V/U.

1.5 Projective spaces. Associated with a vector space V of dimension $(n+1)$ is *n-dimensional projective space* $\mathbb{P}_n = \mathbb{P}(V)$. By the definition, the points of $\mathbb{P}(V)$ are 1-dimensional vector subspaces in V, i.e. the lines in $\mathbb{A}^{n+1} = \mathbb{A}(V)$ passing through the origin. To see them as «usual» points we have to use a screen — an affine hyperplane $U_\xi \subset \mathbb{A}(V)$ that does not contain the origin, like on fig. 1⋄2, and hence given by an affine linear equation $\xi(v) = 1$, where $\xi \in V^* \smallsetminus 0$. Such the screens U_ξ are called *affine charts*. Note that they stay in bijection with non-zero covectors ξ.

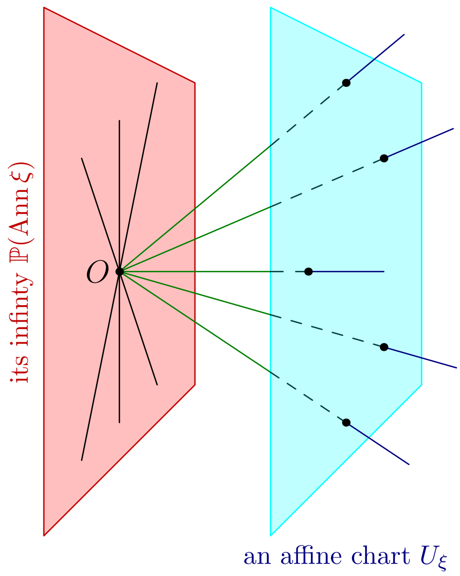

Fig. 1⋄2. Projective word.

No affine chart does cover the whole of $\mathbb{P}(V)$. The difference $\mathbb{P}_n \setminus U_\xi = \mathbb{P}(\operatorname{Ann}\xi) \simeq \mathbb{P}_{n-1}$ consists of all lines lying in the parallel copy of U_ξ drawn through O. It is called *an infinity* of chart U_ξ.

Thus, we have decomposition $\mathbb{P}_n = \mathbb{A}^n \sqcup \mathbb{P}_{n-1}$. Repeating it further, we split $\mathbb{P}_n$ into disjoint union of affine

[1] maybe, an infinite set

[2] i.e. a subspace formed by all finite linear combinations $\sum \lambda_i \varphi_i$ with $\lambda_i \in \Bbbk$

[3] see exrs. 1.8 on p. 8

spaces:

$$\mathbb{P}_n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \mathbb{P}_{n-2} = \cdots = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \ldots \sqcup \mathbb{A}^0 \tag{1-7}$$

(note that $\mathbb{A}^0 = \mathbb{P}_0$ is the one point set).

Exercise 1.15. Consider the decomposition (1-7) over a finite field $\mathbb{F}_q$ of q elements and compute the cardinalities of both sides independently. What the identity on q will you get?

1.5.1 Homogeneous coordinates. A choice of basis $\xi_0, \xi_1, \ldots, \xi_n \in V^*$ identifies V with $\Bbbk^{n+1}$ by sending $v \in V$ to $(\xi_0(v), \xi_1(v), \ldots, \xi_n(v)) \in \Bbbk^{n+1}$. Two non-zero coordinate rows $x, y \in \Bbbk^{n+1}$ represent the same point $p \in \mathbb{P}(V)$ iff they are proportional, i.e. $x_\mu : x_\nu = y_\mu : y_\nu$ for all $0 \leqslant \mu \neq \nu \leqslant n$ (where the identities $0 : x = 0 : y$ and $x : 0 = y : 0$ are allowed as well). Thus, the points $p \in \mathbb{P}(V)$ are in $1-1$ correspondence with collection of ratios $(x_0 : x_1 : \ldots : x_n)$ called *homogeneous coordinates* on $\mathbb{P}(V)$ w.r.t. the chosen basis.

1.5.2 Local affine coordinates. Pick up an affine chart $U_\xi = \{v \in V \mid \xi(v) = 1\}$ on $\mathbb{P}_n = \mathbb{P}(V)$. Any n covectors $\xi_1, \xi_2, \ldots, \xi_n \in V^*$ such that $\xi, \xi_1, \xi_2, \ldots, \xi_n$ form a basis of V^* provide U_ξ with *local affine coordinates*. Namely, consider the basis $e_0, e_1, \ldots, e_m \in V$ dual to $\xi, \xi_1, \xi_2, \ldots, \xi_n$ and let $e_0 \in U_\xi$ be the origin of affine coordinate system and $e_1, e_2, \ldots, e_n \in \operatorname{Ann}\xi$ be its axes. Given a point $p \in \mathbb{P}_n$ with homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$, its local affine coordinates in our system are computed as follows: rescale p to the vector $u_p = p/\xi(p) \in U_\xi$ and evaluate n covectors ξ_ν at u_p to get an n-tiple $x(p) = \big(x_1(p), x_2(p), \ldots, x_n(p)\big)$ in which $x_i(p) \overset{\text{def}}{=} \xi_i(u_p) = \xi_i(p)/\xi(p)$. Note that local affine coordinates are *non-linear* functions of the homogeneous ones.

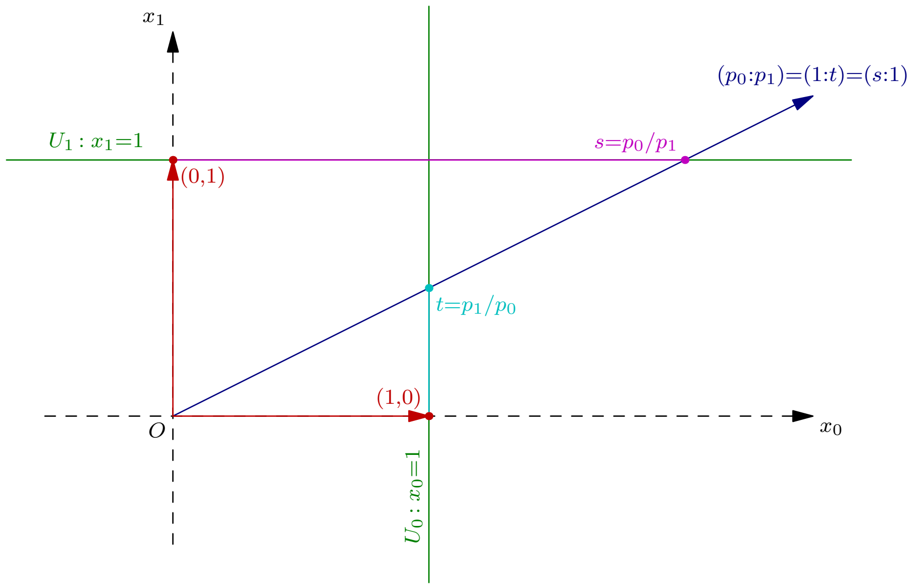

Рис. 1◇3. The standard charts on $\mathbb{P}_1$.

Example 1.6 (projective line)
Projective line $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ is covered by two affine charts $U_0 = U_{x_0}$ and $U_1 = U_{x_1}$, which are the lines in $\mathbb{A}^2 = \mathbb{A}(\Bbbk^2)$ given by equations $x_0 = 1$ and $x_1 = 1$ (see. fig. 1◇3). The chart U_0 covers the whole of $\mathbb{P}_1$ except for one point $(0 : 1)$ corresponding to the vertical coordinate axis in $\Bbbk^2$. A point $(x_0 : x_1)$ with $x_0 \neq 0$ is visible in U_0 as $\left(1 : \frac{x_1}{x_0}\right)$. Function $t = x_1|_{U_0} = x_1/x_0$ can be taken as local affine coordinate in U_0.

Similarly, the chart U_1 covers all points $(x_0 : x_1) = \left(\frac{x_0}{x_1} : 1\right)$ with $x_1 \neq 0$ and $s = x_0|_{U_1} = x_0/x_1$ can be used as local affine coordinate in U_1. The infinite point of U_1 is $(1 : 0)$ corresponding to the horizontal axis in $\mathbb{k}^2$.

As soon as a point $(x_0 : x_1) \in \mathbb{P}_1$ is visible in the both charts, its local affine coordinates s and t satisfy the relation $s = 1/t$.

Exercise 1.16. Check it.

Thus $\mathbb{P}_1$ is a result of gluing two distinct copies of $\mathbb{A}^1$ (one coordinated by s another — by t) along the complement to the origin by the following rule: a point $s \neq 0$ of the first $\mathbb{A}^1$ is glued with the point $t = 1/s$ of the other.

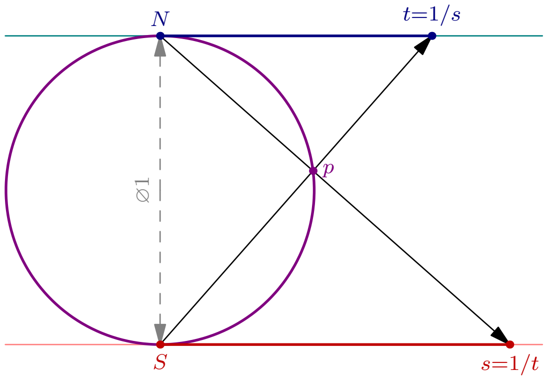

Рис. 1⋄4. $\mathbb{P}_1(\mathbb{R}) \simeq S^1$.

Over $\mathbb{k} = \mathbb{R}$ we get in this way a circle of diameter 1 glued from two opposite tangent lines (see. fig. 1⋄4) via the central projection of each tangent line on the circle from the tangency point of the opposite tangent line.

Similarly, over $\mathbb{k} = \mathbb{C}$ the gluing of two copies of $\mathbb{C}$ by the rule $s \leftrightarrow t = 1/s$ can be realized by means central projections of two tangent planes drown through the south and nord poles of the sphere of diameter 1 onto the sphere from the poles opposite to the tangency poles, see fig. 1⋄5. If we identify each tangent plane with $\mathbb{C}$ respecting their orientations[1] like on fig. 1⋄5, then the complex numbers s, t projected to the point of sphere have opposite arguments and inverse absolute values as we have seen on fig. 1⋄4.

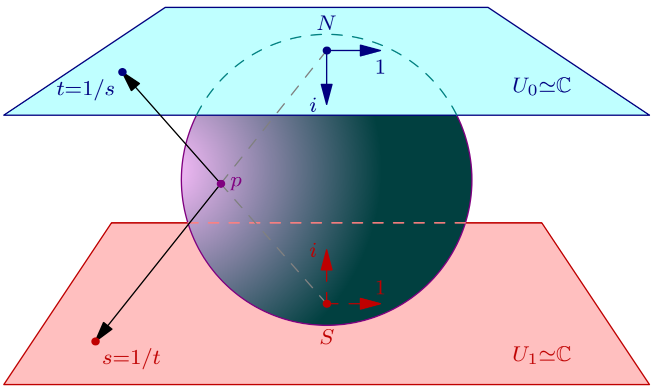

Рис. 1⋄5. $\mathbb{P}_1(\mathbb{C}) \simeq S^2$.

[1]one $\mathbb{C}$ should be obtained from the other by continuous move along the sphere

Exercise 1.17. Make sure that a) the real projective plane $\mathbb{RP}_2$ is the Möbius tape glued with the disc along the boundary circles[1] b) the real projective 3D-space $\mathbb{RP}_3$ coincides with the Lie group $SO_3(\mathbb{R})$ of rotations of the Euclidean space $\mathbb{R}^3$ about the origin.

Example 1.7 (the standard affine covering for $\mathbb{P}_n$)
A collection of $(n+1)$ affine charts $U_\nu = U_{x_\nu}$ given in $\mathbb{k}^{n+1}$ by affine linear equations $\{x_\nu = 1\}$ is called the *standard* affine covering of $\mathbb{P}_n = \mathbb{P}(\mathbb{k}^{n+1})$. For each $\nu = 0, 1, \dots, n$ we take the functions

$$t_i^{(\nu)} = x_i|_{U_\nu} = \frac{x_i}{x_\nu}\,, \quad \text{where } 0 \leqslant i \leqslant n\,,\ i \neq \nu\,,$$

as n standard local affine coordinates inside U_ν.

One can think of $\mathbb{P}_n$ as the result of gluing $(n+1)$ distinct copies $U_0, U_1, \dots, U_n$ of affine space $\mathbb{A}^n$ along their actual intersections inside $\mathbb{P}_n$. In terms of the homogeneous coordinates $x = (x_0 : x_1 : \ \dots\ : x_n)$ on $\mathbb{P}_n$, the intersection $U_\mu \cap U_\nu$ consists of all x with $x_\mu \neq 0$ and $x_\nu \neq 0$. In terms of local affine coordinates inside U_μ and U_ν this intersection is given by inequalities $t_\nu^{(\mu)} \neq 0$ and $t_\mu^{(\nu)} \neq 0$ respectively. Two points $t^{(\mu)} \in U_\mu$ and $t^{(\nu)} \in U_\nu$ are glued with each other in $\mathbb{P}_n$ iff $t_\nu^{(\mu)} = 1/t_\mu^{(\nu)}$ and $t_i^{(\mu)} = t_i^{(\nu)}/t_\mu^{(\nu)}$ for $i \neq \mu, \nu$. RHS of these relations are called *transition functions* from local coordinates $t^{(\nu)}$ to local coordinates $t^{(\mu)}$.

1.6 Projective algebraic varieties. If a basis $x_0, x_1, \dots, x_n \in V^*$ is chosen, non-constant polynomials in x_i's do not produce the functions on $\mathbb{P}(V)$ any more, because the values $f(v)$ and $f(\lambda v)$ are different in general. However for any *homogeneous* polynomial $f \in S^d V^*$ its zero set $V(f) \overset{\text{def}}{=} \{\, v \in V \mid f(v) = 0 \,\}$ is still well defined as a figure in $\mathbb{P}(V)$, because

$$f(v) = 0 \iff f(\lambda v) = \lambda^d f(v) = 0\,.$$

In other words, affine hypersurface $V(f) \subset \mathbb{A}(V)$ defined by homogeneous f is a cone ruled by lines passing through the origin. The set of these lines is denoted by $V(f) \subset \mathbb{P}(V)$ as well and is called *a projective hypersurface* of degree d. Intersections of such hypersurfaces[2] are called *projective algebraic varieties.*

The simplest examples of projective varieties are *projective subspaces* $\mathbb{P}(U) \subset \mathbb{P}(V)$ associated with vector subspaces $U \subset V$ — and given by systems of linear homogeneous equations $\langle \varphi, v \rangle = 0$, where φ runs through $\operatorname{Ann} U$. Say, a line (ab) is associated with the linear span of a, b and consists of points $\lambda a + \mu b$. It could be given by linear equations $\xi(x) = 0$ with ξ running through $\operatorname{Ann}(a) \cap \operatorname{Ann}(b)$ or any set of covectors spanning this space. The ratio $(\lambda : \mu)$ of the coefficients in $\lambda a + \mu b \in (a, b)$ can be taken as internal homogeneous coordinate on the line (ab).

Example 1.8 (smooth affine conics)
On the real projective plane $\mathbb{P}(\mathbb{R}^3)$ consider a curve given by homogeneous equation

$$x_0^2 + x_1^2 = x_2^2 \tag{1-8}$$

and look at its imprints in several affine charts. In the standard chart U_{x_1}, where $x_1 = 1$, in local affine coordinates $t_0 = x_0|_{U_{x_1}} = x_0/x_1$, $t_2 = x_2|_{U_{x_1}} = x_2/x_1$ equation (1-8) turns to hyperbola

$$t_2^2 - t_0^2 = 1\,.$$

[1] the boundary of the Möbius tape is a circle as well as the boundary of the disc
[2] maybe infinite collections of hypersurfaces of different degrees

In other standard chart U_{x_2}, where $x_2 = 1$, in coordinates

$$t_0 = x_0|_{U_{x_2}} = x_0/x_2$$
$$t_1 = x_1|_{U_{x_2}} = x_1/x_2$$

it turns to circle $t_0^2 + t_1^2 = 1$. In slanted chart $U_{x_1+x_2}$, where $x_1 + x_2 = 1$, in local coordinates

$$t = x_0|_{U_{x_1+x_2}} = x_0/(x_1+x_2)$$
$$u = (x_2 - x_1)|_{U_{x_1+x_2}} = (x_2 - x_1)/(x_2 + x_1)$$

we get parabola[1] $t^2 = u$.

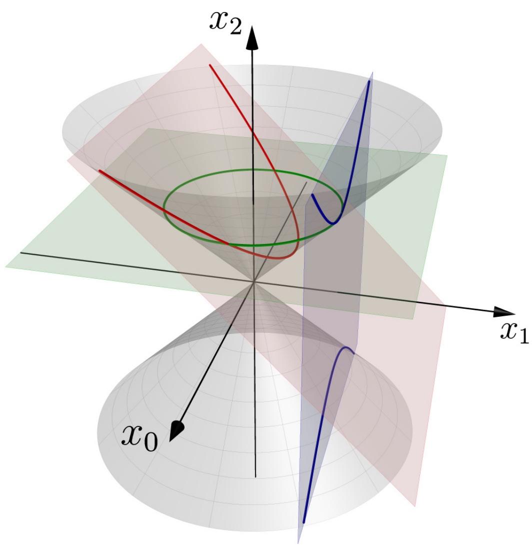

Fig. 1⋄6. The cone.

Thus, affine ellipse, hyperbola and parabola are different pieces of the same projective curve C visible in different affine charts. How does C look like in a given chart $U_\xi \subset \mathbb{P}_2$ depends on positional relationship between C and the infinite line $\ell_\infty = \mathbb{P}(\operatorname{Ann}\xi)$ of the chart: ellipse, hyperbola and parabola appear when ℓ_∞ does not intersect C, does touch C at one point or does intersect C in two distinct points respectively (see. fig. 1⋄6).

1.6.1 Projective closure of an affine variety. Each affine algebraic hypersurface

$$S = V(f) \subset \mathbb{A}^n$$

given by non-homogeneous polynomial $f(x_1, x_2, \dots, x_n)$ of degree d is canonically extent to projective hypersurface $\overline{S} = V(\overline{f}) \subset \mathbb{P}_n$ given by homogeneous polynomial $\overline{f}(x_0, x_1, \dots, x_n) \in S^d V^*$ of the same degree d and such that $\overline{S} \cap U_0 = S$, where $U_0 = U_{x_0}$ is the standard affine chart on $\mathbb{P}_n$. If $f(x_1, x_2, \dots, x_n) = f_0 + f_1(x_1, x_2, \dots, x_n) + f_2(x_1, x_2, \dots, x_n) + \cdots + f_d(x_1, x_2, \dots, x_n)$ where f_i is homogeneous of degree i, then

$$\overline{f}(x_0, x_1, \dots, x_n) = f_0 \cdot x_0^d + f_1(x_1, x_2, \dots, x_n) \cdot x_0^{d-1} + \cdots + f_d(x_1, x_2, \dots, x_n),$$

which turns to f as $x_0 = 1$. The complement $\overline{S} \smallsetminus S = \overline{S} \cap U_0^{(\infty)}$, that is the intersection of $\overline{S}$ with the infinite hyperplane $x_0 = 0$, is given in the homogeneous coordinates $(x_1 : x_2 : \cdots : x_n)$ on the infinite hyperplane by equation $f_d(x_1, x_2, \dots, x_n) = 0$, that is by vanishing the top degree component of f. Thus, infinite points of $\overline{S}$ are nothing else than *asymptotic directions* of affine hypersurface S.

For example, the projective closure of affine cubic curve $x_1 = x_2^3$ is projective cubic $x_0^2 x_1 = x_2^3$ that has exactly one infinite point $p_\infty = (0 : 1 : 0)$. Note that in the standard chart U_1 containing this point C looks like semi-cubic parabola $x_0^2 = x_2^3$ with the cusp at p_∞.

1.6.2 Space of hypersurfaces. Since proportional polynomials define the same hypersurfaces $f = 0$ and $\lambda f = 0$, projective hypersurfaces of fixed degree d are the points of projective space $\mathcal{S}_d = \mathcal{S}_d(V) \stackrel{\text{def}}{=} \mathbb{P}(S^d V^*)$ called *the space of degree d hypersufaces* in $\mathbb{P}(V)$.

Exercise 1.18. Find $\dim \mathcal{S}_d(V)$ assuming $\dim V = n + 1$.

[1] move x_1^2 to R.H.S. of (1-8) and divide the both sides by $x_2 + x_1$

For a fixed point $p \in \mathbb{P}_n = \mathbb{P}(V)$ the equation $f(p) = 0$ is *linear in* $f \in S^d V^*$. Thus, degree d hypersurfaces passing through a given point form a hyperplane in $\mathcal{S}_d$.

Projective subspaces of $\mathcal{S}_d$ are called *linear systems*[1] of hypersurfaces. Any hypersurface of a linear system spanned by

$$V(f_1),\ V(f_2),\ \dots\ ,\ V(f_m)\,,$$

is given by equation $\lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_m f_m = 0$, where $\lambda_1, \lambda_2, \dots, \lambda_m \in \Bbbk$ are some constants. In particular, any such a hypersurface contains the intersection

$$V(f_1) \cap V(f_2) \cap\ \dots\ \cap V(f_m)\,.$$

Traditionally, the linear systems of dimensions 1, 2 and 3 are called *pencils*, *nets* and *webs*.

Exercise 1.19. Show that each pencil of hypersurfaces contains a hypersurface passing through any initially given point (over an arbitrary field $\Bbbk$).

1.6.3 Working example: collections of points on $\mathbb{P}_1$. Let $U = \Bbbk^2$ with the standard coordinates x_0, x_1. Each finite set of points[2] $p_1, p_2, \dots, p_d \in \mathbb{P}_1 = \mathbb{P}(U)$ is the set of zeros for a unique up a scalar factor homogeneous polynomial of degree d

$$f(x_0, x_1) = \prod_{\nu=1}^{d} \det(x, p_\nu) = \prod_{\nu=1}^{d} (p_{\nu,1}x_0 - p_{\nu,0}x_1)\,, \quad \text{where } p_\nu = (p_{\nu,0} : p_{\nu,1})\,. \tag{1-9}$$

We will say that the points p_i are *the roots* of f. Each non-zero homogeneous polynomial of degree d has at most d distinct roots on $\mathbb{P}_1$. If the ground field $\Bbbk$ is algebraically closed, the number of roots[3] equals d precisely and there is a bijection between the points of $\mathbb{P}(S^d U^*)$ and non-ordered collections of d points on $\mathbb{P}_1$.

Over an arbitrary field $\Bbbk$ those collections where all d points coincide with each other form a curve

$$C_d \subset \mathbb{P}_d = \mathbb{P}(S^d U^*)$$

called *the Veronese curve*[4] of degree d. It coincides with an image of *the Veronese embedding*

$$v_d : \mathbb{P}_1^\times = \mathbb{P}\left(U^*\right) \xrightarrow{\varphi \mapsto \varphi^d} \mathbb{P}_d = \mathbb{P}\left(S^d U^*\right) \tag{1-10}$$

that takes a linear polynomial $\varphi \in U^*$, whose zero set is some point $p \in \mathbb{P}(U)$, to d th power $\varphi^d \in S^d(U^*)$, whose zero set is d-tiple point p.

Let us write polynomials $\varphi \in U^*$ and $f \in S^d(U^*)$ as

$$\varphi(x) = \alpha_0 x_0 + \alpha_1 x_1 \quad \text{and} \quad f(x) = \sum_\nu a_\nu \cdot \binom{d}{\nu} x_0^{d-\nu} x_1^\nu$$

and use $(\alpha_0 : \alpha_1)$ and $(a_0 : a_1 : \ \dots\ : a_d)$ homogeneous coordinates in $\mathbb{P}_1^\times = \mathbb{P}(U^*)$ and in $\mathbb{P}_d = \mathbb{P}(S^d U^*)$ respectively. Then the Veronese curve comes with the parametrization

$$(\alpha_0 : \alpha_1) \longmapsto (a_0 : a_1 : \ \dots\ : a_d) = \left(\alpha_0^d : \alpha_0^{d-1}\alpha_1 : \alpha_0^{d-2}\alpha_1^2 : \cdots : \alpha_1^d\right) \tag{1-11}$$

[1] or *linear series* in old terminology

[2] some of points may coincide

[3] counted with *multiplicities*, where a multiplicity of a root p is defined as maximal k such that $\det^k(x, p)$ divides f

[4] there are several other names: *rational normal curve*, *twisted rational curve of degree d* e.t.c.

by the points of $\mathbb{P}_1$. It follows from (1-11) that C_d consists of all $(a_0 : a_1 : \dots : a_d) \in \mathbb{P}_d$ that form a geometric progression, i.e. such that the rows of matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{d-2} & a_{d-1} \\ a_1 & a_2 & a_3 & \dots & a_{d-1} & a_d \end{pmatrix}$$

are proportional. The condition $\operatorname{rk} A = 1$ is equivalent to vanishing of all 2×2-minors of A. Thus, $C_d \subset \mathbb{P}_d$ is given by a system of quadratic equations.

An intersection of C_d with an arbitrary hyperplane given by equation

$$A_0 a_0 + A_1 a_1 + \dots + A_d a_d = 0 ,$$

consists of the Veronese-images of the roots $(\alpha_0 : \alpha_1) \in \mathbb{P}_1$ of homogeneous polynomial

$$\sum_\nu A_\nu \cdot \alpha_0^{d-\nu} \alpha_1^\nu$$

of degree d. Since it has at most d roots, any $d+1$ distinct points on the Veronese curve do not lie in a hyperplane. This implies that any m points of C_d span a subspace of dimension $m+1$ and do not lie in a common subspace of dimension $(m-2)$ as soon $2 \leqslant m \leqslant d+1$.

If $\Bbbk$ is algebraically closed, C_d intersects any hyperplane in precisely d points (some of which may coincide). This explains why we did say that C_d has degree d.

Example 1.9 (Veronese conic)
The Veronese conic $C_2 \subset \mathbb{P}_2$ consists of quadratic trinomials

$$a_0 x_0^2 + 2a_1 x_0 x_1 + a_2 x_1^2 ,$$

that are perfect squares of linear forms. It is given by well known equation

$$D/4 = -\det \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} = a_1^2 - a_0 a_2 = 0 \tag{1-12}$$

and comes with rational parametrization

$$a_0 = \alpha_0^2 , \quad a_1 = \alpha_0 \alpha_1 , \quad a_2 = \alpha_1^2 . \tag{1-13}$$

1.7 Subspaces and projections. Projective subspaces $K = \mathbb{P}(U)$ and $L = \mathbb{P}(W)$ in $\mathbb{P}_n = \mathbb{P}(V)$ are called *complementary*, if $K \cap L = \varnothing$ and $\dim K + \dim L = n - 1$. For example, any two non-intersecting lines in $\mathbb{P}_3$ are complementary. In terms of linear algebra, the vector subspaces $U, W \subset V$ have zero intersection $U \cap V = \{0\}$ and

$$\dim U + \dim W = \dim K + 1 + \dim L + 1 = (n+1) = \dim V .$$

Thus, $V = U \oplus W$ and $v \in V$ has a unique decomposition $v = u + w$ where $u \in U$ and $w \in W$. Since the both components u, w are non vectors as soon v dos not belong neither U nor W, we conclude that each point $p \notin K \sqcup L$ lies on a unique line intersecting both subspaces K, L.

Exercise 1.20. Make it sure.

Given a pair of complementary subspaces $K, L \subset \mathbb{P}_n$, a *projection from K to L* is a map

$$\pi_L^K : (\mathbb{P}_n \setminus K) \to L$$

that sends each point $p \in \mathbb{P}_n \setminus (K \sqcup L)$ to a unique point $b \in L$ such that line pb intersects K and sends each point of L to itself. In homogeneous coordinates $(x_0 : x_1 : \ \dots \ : x_n)$ such that $(x_0 : x_1 : \ \dots \ : x_m)$ are the coordinates in K and $(x_{m+1} : x_{m+2} : \ \dots \ : x_n)$ are the coordinates in L, projection π_L^K removes the first $(m+1)$ coordinates x_ν, $0 \leqslant \nu \leqslant m$.

Example 1.10 (Projecting a conic to a line)
Consider a smooth conic $C \subset \mathbb{P}_2$ given by equation[1] $x_0^2 + x_1^2 = x_2^2$, a line $L \subset \mathbb{P}_2$ given by equation $x_0 = 0$, a point $p = (1 : 0 : 1) \in C$, and the projection $\pi_L^p : C \to L$ of C to L from p. We extend it into p by the rule $p \mapsto (0 : 1 : 0)$.

In the standard affine chart U_2, where $x_2 = 1$, it looks as fig. 1⋄7 and provides a bijection between L and C. Moreover, it is *birational*, i.e. homogeneous coordinates of the corresponding points $q = (q_0 : q_1 : q_2) \in C$ and $t = (0 : t_1 : t_2) = \pi_L^p(q) \in L$ are *rational* algebraic functions of each other:

$$\begin{aligned} (t_1 : t_2) &= (q_1 : (q_2 - q_0)) \\ (q_0 : q_1 : q_2) &= \\ &= ((t_1^2 - t_2^2) : 2\,t_1 t_2 : (t_1^2 + t_2^2)) \end{aligned} \tag{1-14}$$

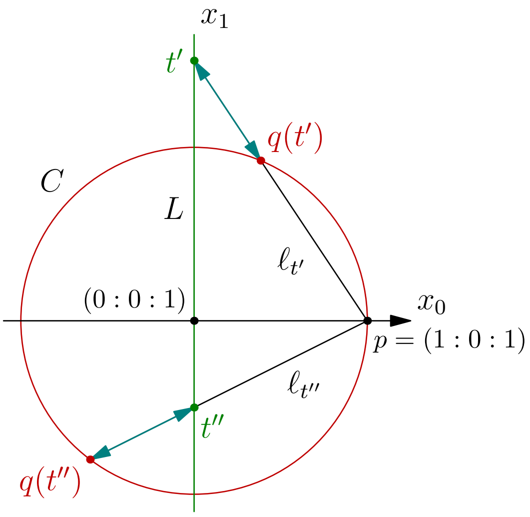

Fig. 1⋄7. Projecting a conic.

Exercise 1.21. Check these formulas and use the second of them to list all integer solutions of the Pythagor equation $a^2 + a^2 = c^2$.

A linear change of homogeneous coordinates by formulas

$$\begin{cases} a_0 = x_2 + x_0 \\ a_1 = x_1 \\ a_2 = x_2 - x_0 \end{cases} \qquad \begin{cases} x_0 = (a_0 - a_2)/2 \\ x_1 = a_1 \\ x_0 = (a_0 + a_2)/2 \end{cases}$$

transforms C to the Veronese conic $a_1^2 = a_0 a_2$ from (1-12) and the standard parametrisation (1-13) of the Veronese curve turns to the parametrisation (1-14).

1.8 Linear projective transformations. Each linear isomorphism $F : U \xrightarrow{\sim} W$ produces well defined bijection $\overline{F} : \mathbb{P}(U) \xrightarrow{\sim} \mathbb{P}(W)$ called *a linear projective isomorphism.*

Exercise 1.22. Given two hyperplanes L_1, $L_2 \subset \mathbb{P}_n = \mathbb{P}(V)$ and a point $p \notin L_1 \cup L_2$, check that a projection from p to L_2 induces linear projective isomorphism$\gamma_p : L_1 \xrightarrow{\sim} L_2$.

Lemma 1.2
For any 2 ordered collections of $(n+2)$ points $\{p_0, p_1, \dots, p_{n+1}\} \in \mathbb{P}(U)$, $\{q_0, q_1, \dots, q_{n+1}\} \in \mathbb{P}(W)$ such that no $(n+1)$ points of each lie in a hyperplane there exists a unique up a scalar factor linear projective isomorphism $F : U \xrightarrow{\sim} W$ taking $\overline{F}(p_i) = q_i$ for all i.

Proof. Fix some vectors u_i and w_i representing the points p_i and q_i and pick up $\{u_0, u_1, \dots, u_n\}$ and $\{w_0, w_1, \dots, w_n\}$ as the bases for U and W. A linear mapping $F : U \to W$ sends $p_i \mapsto q_i$ iff

[1]the same as in example 1.8

$F(u_i) = \lambda_i w_i$ for some non-zero $\lambda_i \in \Bbbk$. Thus, a matrix of F in the chosen bases is the diagonal matrix $\operatorname{diag}(\lambda_0, \lambda_1, \dots, \lambda_n)$. Further, all coordinates x_i in expansion of $(n+1)$ th vector

$$u_{n+1} = x_0 u_0 + x_1 u_1 + \dots + x_n u_n$$

are non-zero, because otherwise $n+1$ point: p_{n+1} and all p_i's whose number differs from the number of vanishing coordinate turn lying in the same coordinate hyperplane. If

$$w_{n+1} = y_0 w_0 + y_1 w_1 + \dots + y_n w_n\,,$$

then the condition $F(u_{n+1}) = \lambda_{n+1} w_{n+1}$ means that $y_i = \lambda_{n+1}\lambda_i x_i$ for all $0 \leqslant i \leqslant n$. This fixes $(\lambda_0, \lambda_1, \dots, \lambda_n) = \lambda_{n+1}^{-1} \cdot \big(y_1/x_1,\, y_2/x_2,\, \dots,\, y_n/x_n\big)$ uniquely up to a scalar factor $\lambda_{n+1}^{-1} \neq 0$. □

Corollary 1.1
Two non-degenerated matrices produce the same linear projective isomorphism iff they are proportional. □

1.8.1 Projective linear group. Linear projective automorphisms of $\mathbb{P}(V)$ form a group called *projective linear group* and denoted $\mathrm{PGL}(V)$. It follows from lemma 1.2 that this group is isomorphic to the factor group of the linear group $\mathrm{GL}(V)$ by the subgroup $H = \{\lambda \cdot \mathrm{Id} \mid \lambda \neq 0\}$ of the scalar dilatations: $\mathrm{PGL}(V) = \mathrm{GL}(V)/H$. A choice of basis in V identifies $\mathrm{GL}(V)$ with the group $\mathrm{GL}_{n+1}(\Bbbk)$ of non-degenerated matrices. Then $\mathrm{PGL}(V)$ is identified with $\mathrm{PGL}_{n+1}(\Bbbk)$, which is «a projectivisatio» of GL_{n+1}, that is, non-degenerated matrices up to rescaling.

Example 1.11 (linear fractional transformations)
For $n = 1$ we get group $\mathrm{PGL}_2(\Bbbk)$ consisting of non-degenerated 2×2-matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc \neq 0$$

up to a constant factor. It acts on $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ by the standard rule

$$\overline{A} : (x_0 : x_1) \longmapsto \big((ax_0 + bx_1) : (cx_0 + dx_1)\big)\,.$$

In affine chart $U_1 \simeq \mathbb{A}^1$ with local coordinate $t = x_0/x_1$ this action looks like linear fractional transformation

$$t \longmapsto \frac{at + b}{ct + d}$$

which does not change under rescaling of matrix, certainly. It is also clear that a linear fractional map taking 3 distinct points q, r, s to ∞, 0, 1 respectively is forced to be

$$t \longmapsto \frac{t - r}{t - q} \cdot \frac{s - r}{s - q} \tag{1-15}$$

in agreement with lemma 1.2.

1.8.2 Homographies. Linear fractional isomorphisms between two distinct projective lines are called *homographies*.

Lemma 1.3
Let $\Bbbk$ be algebraically closed and $\varphi : \mathbb{P}_1(\Bbbk) \smallsetminus \{\text{finite set of points}\} \xrightarrow{\sim} \mathbb{P}_1(\Bbbk) \smallsetminus \{\text{finite set of points}\}$, be a bijection given in some affine chart with a local coordinate t as

$$t \mapsto \varphi(t) = g(t)/h(t)\,, \quad \text{where} \quad g, h \in \Bbbk[t]\,. \tag{1-16}$$

Then φ is a linear fractional map, that is, a homography[1]

Proof. In homogeneous coordinates $(x_0 : x_1)$ such that $t = x_0/x_1$ we can rewrite[2] formula (1-16) for φ as $(x_0 : x_1) \mapsto \big(F(x_0, x_1) : G(x_0, x_1)\big)$, where F and G are homogeneous polynomials of the same degree $d = \deg F = \deg G$ in (x_0, x_1) without common zeros (in particular, F and G are non-proportional). Write $\mathbb{P}_d$ for the projectivization of the space of homogeneous degree d polynomials in (x_0, x_1). As soon as a pont $\vartheta = (\vartheta_0 : \vartheta_1) \in \mathbb{P}_1$ has precisely one pre-image under φ a polynomial

$$H_\vartheta(x_0, x_1) = \vartheta_1 \cdot F(x_0, x_1) - \vartheta_0 \cdot G(x_0, x_1)$$

has precisely one root $x = \varphi^{-1}(\vartheta)$ on $\mathbb{P}_1$. Since $\Bbbk$ is algebraically closed, this root has to be of multiplicity d. Thus, H_ϑ is pure d th power of a linear form, that is, H_ϑ lies on the Veronese curve $C_d \subset \mathbb{P}_d$ from n° 1.6.3 on p. 16. On the other hand, H_ϑ runs through the line $(F, G) \subset \mathbb{P}_d$ when ϑ runs through $\mathbb{P}_1$. Since $\mathbb{P}_1(\Bbbk)$ is infinite, we get infinite intersection set $(FG) \cap C_d$. But we have seen in n° 1.6.3 for $d \geqslant 2$ any 3 points of C_d are non-collinear. Hence $d = 1$ and $\varphi \in \mathrm{PGL}_2(\Bbbk)$. □

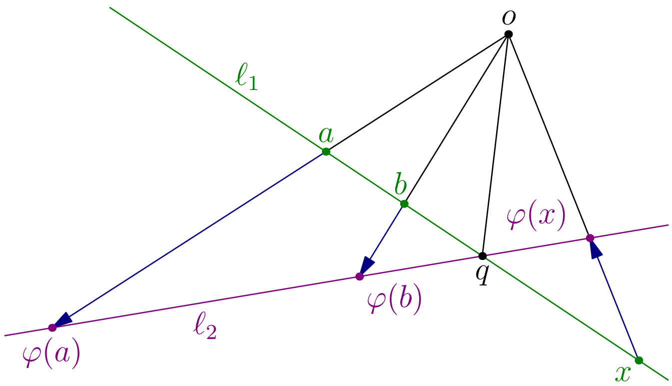

Рис. 1⋄8. Perspective $o : \ell_1 \xrightarrow{\sim} \ell_2$.

Example 1.12 (perspectives)
The simplest example of homography is a projection of a line $\ell_1 \subset \mathbb{P}_2$ to another line $\ell_2 \subset \mathbb{P}_2$ from an arbitrary point $o \notin \ell_1 \cup \ell_2$ (see. fig. 1⋄8). We will call it *a perspective with center* o and denote $o : \ell_1 \xrightarrow{\sim} \ell_2$.

Exercise 1.23. Make sure that a perspective is a homography.

An arbitrary homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ is a perspective iff it sends the intersection point $\ell_1 \cap \ell_2$ to itself. Indeed, pick up any two distinct points $a, b \in \ell_1 \smallsetminus \ell_2$ and put $o = (a, \varphi(a)) \cap (b, \varphi(b))$

[1] in particular, φ has a unique extension to the whole of $\mathbb{P}_1$
[2] perhaps, after some modification of the finite set where φ is undefined

(see fig. 1⋄8). Then the perspective $o : \ell_1 \rightleftharpoons \ell_2$ sends an ordered triple of points a, b, $\ell_1 \cap \ell_2$ to $\varphi(a)$, $\varphi(b)$, $\ell_1 \cap \ell_2$. Thus, it coincides with φ iff φ keeps the intersection of lines fixed.

Proposition 1.6
Let $\ell_1, \ell_2 \subset \mathbb{P}_2$ and $q = \ell_1 \cap \ell_2$. For any line ℓ composition of consequent perspectives

$$\varphi = \left(b_1 : \ell \to \ell_2\right) \circ \left(b_2 : \ell_1 \to \ell\right), \quad \text{where} \quad b_1 \in \ell_1,\ b_2 \in \ell_2, \tag{1-17}$$

takes $b_1 \mapsto b_2$ and has $\varphi(q), \varphi^{-1}(q) \in \ell$. Each homography $\varphi : \ell_1 \rightleftharpoons \ell_2$ can be represented as some composition (1-17), where $b_1 \in \ell_1$ can be chosen arbitrarily, $b_2 = \varphi(b_1)$, and ℓ *does not depend* on a choice of $b_1 \in \ell_1$.

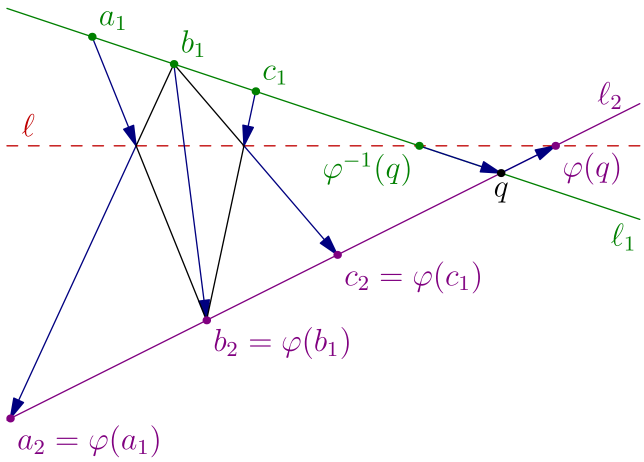

Рис. 1⋄9. Cross-axis.

Proof. The first assertion is clear from fig. 1⋄9. To prove the second, pick up a triple of distinct points $a_1, b_1, c_1 \in \ell_1 \smallsetminus \{q\}$ and write $a_2, b_2, c_2 \in \ell_2$ for their images under φ. Take ℓ to be the line joining intersections of two pairs of «cross-lines» $\left(a_1 b_2\right) \cap \left(b_1 a_2\right)$ and $\left(c_1 b_2\right) \cap \left(b_1 c_2\right)$. Then fig. 1⋄9 shows that the composition in R.H.S. of (1-17) sends a_1, b_1, c_1 to a_2, b_2, c_2 respectively. Hence, it coincides with φ.

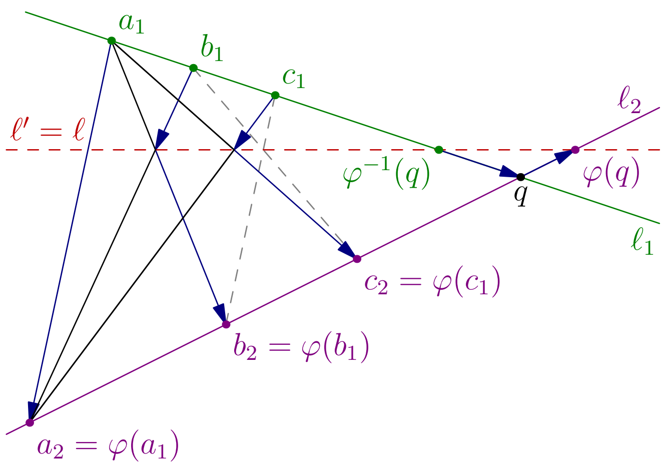

Рис. 1⋄10. Coincidence $\ell' = \ell$.

To check that ℓ does not depend on b_1 let us repeat the arguments for ordered triple c_1, a_1, b_1 instead of a_1, b_1, c_1 (see fig. 1⋄10). We get the decomposition $\varphi = (a_1 : \ell' \to \ell_2) \circ (a_2 : \ell' \to \ell)$, where ℓ' joins the cross-intersections $(a_1c_2) \cap (c_1a_2)$ and $(b_1a_2) \cap (a_1, b_2)$. Since both lines ℓ, ℓ' pass through $(b_1a_2) \cap (a_1, b_2)$, $\varphi(q)$, $\varphi^{-1}(q)$ (latter two coincide if φ is a perspective), they coincide: $\ell = \ell'$. It implies that *all* cross-intersections $(x, \varphi(y)) \cap (y, \varphi(x))$, where $x \neq y$ are running through ℓ_1, *lie on the same line* ℓ. This property characterizes ℓ uniquely. □

Definition 1.2 (cross-axis of a homography)
Given a homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$, the line ℓ drown by the cross-intersections $(x, \varphi(y)) \cap (y, \varphi(x))$ when $x \neq y$ run through ℓ_1 is called the *cross-axis* of the homography φ.

Remark 1.1. The cross-axis has to pass through $\varphi\left(\ell_1 \cap \ell_2\right)$ and $\varphi^{-1}\left(\ell_1 \cap \ell_2\right)$. If φ is a perspective and $\varphi\left(\ell_1 \cap \ell_2\right) = \varphi^{-1}\left(\ell_1 \cap \ell_2\right) = \ell_1 \cap \ell_2$, then the cross-axis can not be recovered from the action of φ just on the intersection of the lines. However prop. 1.6 (and its proof) hold in this case as well.

Exercise 1.24. Let a homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ send 3 given points $a_1, b_1, c_1 \in \ell_1$ to 3 given points $a_2, b_2, c_2 \in \ell_2$. Using only the ruler, construct $\varphi(x)$ for a given $x \in \ell_1$.

1.9 Cross-ratio. Consider $\mathbb{P}_1 = \mathbb{P}(\Bbbk^2)$ with the standard homogeneous coordinates $(x_0 : x_1)$ and put $x = x_0/x_1$. Then for any two points $a = (a_0 : a_1)$ and $b = (b_0 : b_1)$ the difference of their affine coordinates $a = a_0/a_1$ and $b = b_0/b_1$ coincides up to a scalar factor with the determinant of their homogeneous coordinates:

$$a - b = \frac{a_0}{a_1} - \frac{b_0}{b_1} = \frac{a_0b_1 - a_1b_0}{a_1b_1} = \frac{\det(a, b)}{a_1b_1}\,.$$

Definition 1.3
Given 4 distinct points $p_1, p_2, p_3, p_4 \in \mathbb{P}_1$, their *cross-ratio* is

$$[p_1, p_2, p_3, p_4] \overset{\text{def}}{=} \frac{(p_1 - p_3)(p_2 - p_4)}{(p_1 - p_4)(p_2 - p_3)} = \frac{\det(p_1, p_3) \cdot \det(p_2, p_4)}{\det(p_1, p_4) \cdot \det(p_2, p_3)}\,. \tag{1-18}$$

1.9.1 Geometrical meaning of the cross-ratio. It follows from (1-15) that $[p_1, p_2, p_3, p_4]$ coincides with the image of p_4 under those unique homography that sends p_1, p_2, p_3 to ∞, 0, 1 respectively. Hence, it can take any value except for ∞, 0, 1 and an ordered quadruple of distinct points can be moved to another such a quadruple by a homography iff their cross-ratios coincide.

Exercise 1.25. Prove the latter statement.

Since a linear change of coordinates can be viewed as a homography, R.H.S. of (1-18) does not depend on a choice of coordinates, and the middle part of (1-18) does not depend either on a choice of affine chart containing the points[1] or on a choice of local affine coordinate.

[1]algebraically, this means that all values p_1, p_2, p_3, p_4 are finite

1.9.2 The action of $\mathfrak{S}_4$. Symmetric group $\mathfrak{S}_4$ acts on a given quadruple of points by permutations. It contains the normal subgroup of Klein $\mathfrak{D}_2 \subset \mathfrak{S}_4$, which consists of the identity map and 3 pairs of independent transpositions:

$$(2,\ 1,\ 4,\ 3)\,,\quad (3,\ 4,\ 1,\ 2)\,,\quad (4,\ 3,\ 2,\ 1) \tag{1-19}$$

. It is clear that $\mathfrak{D}_2$ does not change the cross-ratio:

$$[p_1,\, p_2,\, p_3,\, p_4] = \frac{\det\left(p_1, p_3\right)\cdot \det\left(p_2, p_4\right)}{\det\left(p_1, p_4\right)\cdot \det\left(p_2, p_3\right)} = \\ = [p_2,\, p_1,\, p_4,\, p_3] = [p_3,\, p_4,\, p_1,\, p_2] = [p_4,\, p_3,\, p_2,\, p_1]\,.$$

Thus, the action of $\mathfrak{S}_4$ on $\vartheta = [p_1,\, p_2,\, p_3,\, p_4]$ is factorized through the epimorphism

$$\mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_3 \simeq \mathfrak{S}_4/\mathfrak{D}_2 \tag{1-20}$$

whose kernel is the Klein subgroup $\mathfrak{D}_2$. Geometrically, surjection (1-20) is visible as follows.

In Euclidean space $\mathbb{R}^3$ consider a cube centred at the origin and number its internal diagonals by $1, 2, 3, 4$ (see fig. 1◇11). The group of all rotations of $\mathbb{R}^3$ about the origin sending the cube to itself is called the (*proper*) *group of cube*. It permutes the diagonals. Next exercise:

Exercise 1.26. Check that a) a rotation that sends each the diagonal to itself coincides with identity $\mathrm{Id}_{\mathbb{R}^3}$ b) any two diagonals can be interchanged by a rotation that sends each of two remaining diagonals to itself.

shows that the proper group of cube is isomorphic to $\mathfrak{S}_4$. On the other hand, the group of cube acts on the Cartesian coordinate lines x, y, z, which join the centres of the opposite faces of the cube. This gives homomorphism (1-20). Its kernel consists of 3 rotations by 180° about the axes x, y, z and the identity map. It is nothing than $\mathfrak{D}_2$.

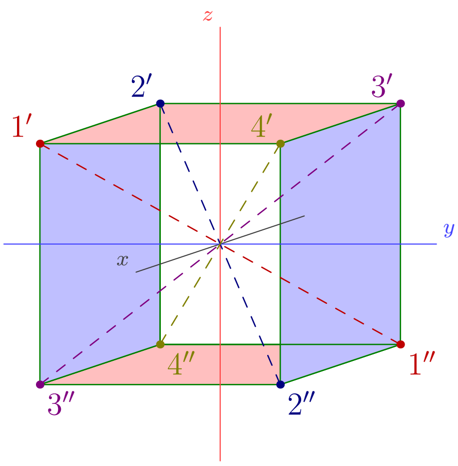

Рис. 1◇11. Group of cube.

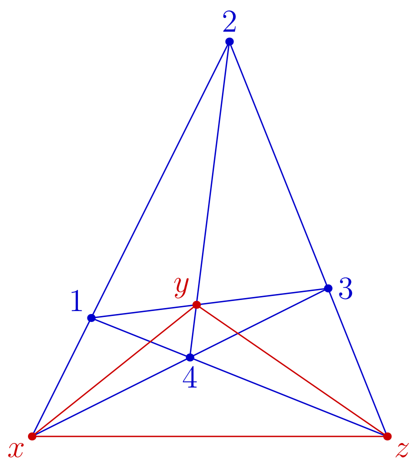

Рис. 1◇12. Complete quadrangle.

On $\mathbb{P}_2 = \mathbb{P}(\mathbb{R}^3)$ the diagonals of the cube are looking as 4 pints without collinear triples among them. The planes spanned by the diagonals form «sides» and «diagonals» of a figure called *a completed quadrangle* (see fig. 1◇12). Besides 4 vertexes $1, 2, 3, 4$ of the quadrangle, its diagonals are intersecting at 3 more points x , y , z , which visualise the Cartesian coordinate lines of $\mathbb{R}^3$. They form what is called an *associated triangle* of the quadrangle 1, 2, 3, 4.

Group $\mathfrak{S}_4$ renumbers the vertices of the quadrangle. Induced permutations of vertices

$$x = (12) \cap (34)\,, \quad y = (13) \cap (24)\,, \quad z = (14) \cap (23)$$

produces surjection (1-20) with kernel (1-19). Group $\mathfrak{S}_3$ being identified with *the group the triangle* xyz consists of 3 «reflections» (yz) , (xz) , (xy) , which are the left cosets of $\mathfrak{V}_2$ in $\mathfrak{S}_4$ spanned respectively by the transpositions $(1, 2)$, $(1, 3)$, $(2, 3)$ of $\mathfrak{S}_4$, and 3 «rotations», which are the the left cosets of $\mathfrak{V}_2$ in $\mathfrak{S}_4$ spanned by the identity and the cycles $(1, 2, 3)$ and $(1, 3, 2)$.

If $[p_1, p_2, p_3, p_4] = \vartheta$, then Id, $(1, 2)$, $(1, 3)$, $(2, 3)$, $(1, 2, 3)$, $(1, 3, 2)$ take it to

$$\begin{aligned}
&[p_1, p_2, p_3, p_4] = [p_2, p_1, p_4, p_3] = [p_3, p_4, p_2, p_1] = [p_4, p_3, p_2, p_1] = \vartheta\\
&[p_2, p_1, p_3, p_4] = [p_1, p_2, p_4, p_3] = [p_3, p_4, p_1, p_2] = [p_4, p_3, p_1, p_2] = 1/\vartheta\\
&[p_3, p_2, p_1, p_4] = [p_2, p_3, p_4, p_1] = [p_1, p_4, p_2, p_3] = [p_4, p_1, p_2, p_3] = \vartheta/(\vartheta-1)\\
&[p_1, p_3, p_2, p_4] = [p_3, p_1, p_4, p_2] = [p_2, p_4, p_1, p_3] = [p_4, p_2, p_3, p_1] = 1-\vartheta\\
&[p_2, p_3, p_1, p_4] = [p_3, p_2, p_4, p_1] = [p_1, p_4, p_3, p_2] = [p_4, p_1, p_3, p_2] = (\vartheta-1)/\vartheta\\
&[p_3, p_1, p_2, p_4] = [p_1, p_3, p_4, p_2] = [p_2, p_4, p_1, p_3] = [p_4, p_2, p_1, p_3] = 1/(1-\vartheta)\,.
\end{aligned} \tag{1-21}$$

Exercise 1.27. Check this without fail.

1.9.3 Special quadruples of points. It follows from (1-21) that there are precisely 3 special values $\vartheta = -1,\ 2,\ 1/2$ that are not changed by the transpositions $(1, 2)$, $(1, 3)$, $(1, 4)$ respectively. They satisfy the quadratic equations

$$\vartheta = \frac{1}{\vartheta}\,, \quad \vartheta = \frac{\vartheta}{\vartheta - 1} \quad \text{and} \quad \vartheta = 1 - \vartheta$$

and are cyclically permuted by the rotations from the group of triangle. Also there are 2 special values of ϑ that are not changed by the rotations are interchanged by the transpositions from the group of triangle. They satisfy the quadratic equation[1]

$$\vartheta^2 - \vartheta + 1 = 0 \iff \vartheta = \frac{\vartheta - 1}{\vartheta} \iff \vartheta = \frac{1}{1 - \vartheta}\,.$$

We call *special* the 5 values of ϑ just listed. Quadruples of points with special cross-ratios will be also called *special quadruples*. Permuting the points of non-special quadruple, one gets 6 distinct values (1-21). Permutations in a special quadruple lead to either 3 or 2 distinct values.

1.9.4 Harmomic pairs. A special quadruple $\{a, b; c, d\} \in \mathbb{P}_1$ whith $[a, b, c, d] = -1$ is called *harmonic*. Geometrically, this means that b is the middle point of segment $[c, d]$ in an affine piece with the infinity at a.

Algebraically, in terms of (1-21), harmonicity means that the cross-ratio is not changed either by transposition (12), or by transposition (34). Note that each of these two properties forces the quadruple to be harmonic.

Since the order preserving exchange of p_1, p_2 with p_3, p_4 keeps the cross-ratio fixed, we conclude that the harmonicity considered as a relation between pairs $\{p_1, p_2\}$ and $\{p_3, p_4\}$ is a *symmetric relation* on a set of *non-ordered* pairs of points in $\mathbb{P}_1$. This allows to speak about *harmonic pairs* of non-ordered pairs of points or, equivalently, about harmonic pairs of binary quadratic forms.

[1] i.e. different from -1 cubic roots of unity in $\Bbbk$

Example 1.13 (harmonicity in quadrangle)
In $\mathbb{P}_2$ consider an arbitrary quadrangle with vertices a, b, c, d and associated triangle x, y, z (see fig. 1⋄13). Then, for each vertex x, y, z consider a pencil of lines passing through this vertex. We claim that in each pencil the pair of sides of the quadrangle is harmonic to the pair of sides of the triangle.

To verify this at x, let us parametrize the pencil of lines passing through x either by the points of line (ad) or by the points of line (bc) and put

$$x' = (xy) \cap (ad)$$
$$x'' = (xy) \cap (bc)$$

We have to check that

$$[a, d, z, x'] = [b, c, z, x''] = -1 .$$

Since the perspectives with centres at x and at y preserve the cross-ratios, we have

$$[a, d, z, x'] = [b, c, z, x''] = [d, a, z, x']$$

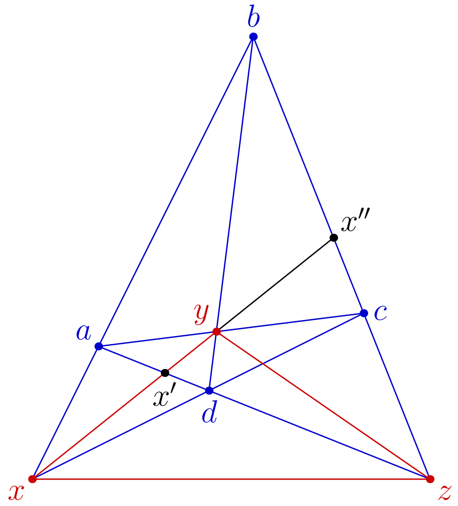

Fig. 1⋄13. Harmonic pairs of sides.

(see fig. 1⋄13). Thus, the transposition of the first two points did not changed the cross-ratio. Hence, the points are harmonic.

Home task problems to §1

Problem 1.1. Let the ground field $\Bbbk$ consist of q elements. Find the total number of k-dimensional
a) vector subspaces in $\Bbbk^n$ b) affine subspaces in $\mathbb{A}^n$ c) projective subspaces in $\mathbb{P}_n$.
(Hint: to begin with, take $k = 0,\ 1,\ 2,\ \dots$)

Problem 1.2. Compute the limits of the previous answers as $q \to 1$.

Problem 1.3. Find a geometric condition on 3 lines ℓ_1, ℓ_2, ℓ_3 in $\mathbb{P}_2 = \mathbb{P}(V)$ necessary and sufficient for existence a coordinate system in V such that each ℓ_i becomes the infinite line for the standard chart $U_i = U_{x_i}$ in these coordinates.

Problem 1.4. Non-identical linear projective automorphism σ is called *an involution*, if $\sigma^2 = \mathrm{Id}$. Over algebraically closed field show that each involution of the projective line has precisely two distinct fixed points.

Problem 1.5. There are two points on a wall and a ruler whose length is significantly shorter than a distance between the points. Draw a straight line joining the points.

Problem 1.6. A point P and two non-parallel lines are drawn on a sheet of paper. The intersection point Q of the lines is far outside the sheet border. Using only the ruler, draw a part of line PQ laying inside the sheet.

Problem 1.7. Put real Euclidian plane $\mathbb{R}^2$ into $\mathbb{CP}_2$ as the real part of the standard chart $U_0 = \mathbb{C}^2$.
a) Find two points $A_\pm \in \mathbb{CP}_2$ laying on all conics visible in $\mathbb{R}^2$ as the circles.
b) Let a conic $C \subset \mathbb{CP}_2$ have at least 3 non-collinear points in $\mathbb{R}^2$ and pass through $A_\pm$. Show that $C \cap \mathbb{R}^2$ is a circle.

Problem 1.8 (the Papus theorem). Let points a_1, b_1, c_1 be collinear and points a_2, b_2, c_2 be collinear as well. Show that triple intersection points $(a_1b_2) \cap (a_2b_1)$, $(b_1c_2) \cap (b_2c_1)$, $(c_1a_2) \cap (c_2a_1)$ is collinear too.

Problem 1.9. Formulate and prove the dual statement[1] to the Papus theorem.

Problem 1.10 (1st theorem of Dezargus). Given 2 triangles $A_1B_1C_1$ and $A_2B_2C_2$ on $\mathbb{P}_2$, show that three intersection points $(A_1B_1) \cap (A_2B_2)$, $(B_1C_1) \cap (B_2C_2)$, $(C_1A_1) \cap (C_2A_2)$ are collinear iff three lines (A_1A_2), (B_1B_2), (C_1C_2) are intersecting at one point[2].

Problem 1.11 (2nd theorem of Dezargus). Let a line ℓ pass through three distinct points p, q, r but do not contain any of three other distinct points a, b, c. Show that lines (ap), (bq), (cr) are intersecting at one point iff there exists an involution of ℓ that exchanges p, q, r with intersection points of ℓ with lines (bc), (ca), (ab) respectively.

Problem 1.12. For any two projective subspaces $K, L \subset \mathbb{P}_n$ prove inequality

$$\dim(K \cap L) \geqslant \dim K + \dim L - n\,.$$

Problem 1.13. Find triple of points $A, B, C \in \mathbb{P}_2$ such that three given points $A' = (1 : 0 : 0)$, $B' = (0 : 1 : 0)$, $C' = (0 : 0 : 1)$ lie on the lines (BC), (CA), (AB) respectively and three lines (AA'), (BB'), (CC') are intersecting at $(1 : 1 : 1)$.

Problem 1.14 (projecting twisted cubic). Let $\mathbb{P}_1 = \mathbb{P}(V^*)$ be the (projectivization of the) space of linear forms in two variables (t_0, t_1), $\mathbb{P}_3 = \mathbb{P}(S^3V^*)$ be the space of cubic forms in (t_0, t_1), and $C_3 = \{\varphi^3 \mid \varphi \in V^*\} \subset \mathbb{P}_3$ be the subspace of pure cubes. Describe a projection of C_3
a) from the point t_0^3 to the plane spanned by $3\,t_0^2t_1$, $3\,t_0t_1^2$, and t_1^3
b) from the point $3\,t_0^2t_1$ to the plane spanned by t_0^3, $3\,t_0t_1^2$, and t_1^3
c) from the point $t_0^3 + t_1^3$ to the plane spanned by t_0^3, $3\,t_0^2t_1$, and $3\,t_0t_1^2$
More precisely, write explicit parametric representation for the projection in appropriate coordinates, then find its affine and homogeneous equation. Do that for several affine charts on the target plane of the projection. In each case, find degree of the curve and try to draw it. Has it self-intersections and/or cusps?

[1] that holds in the dual space $\mathbb{P}_2^\times = \mathbb{P}(V^*)$ and dials with the annihilators of all the subspaces from the original statement

[2] pair of triangles with these properties is called *perspective*

§2 Projective Quadrics

During §2 we always assume that $\operatorname{char} \Bbbk \neq 2$.

2.1 Reminders from linear algebra. Projective hypersurfaces of degree 2 are called *projective quadrics*. We write $Q \subset \mathbb{P}(V)$ for quadric $Q = V(q)$ given as the zero set of non-zero quadratic form $q \in S^2V^*$.

If $2 \neq 0$ in $\Bbbk$, each $q \in S^2V^*$ can be written as $q(x) = \sum a_{ij}\, x_i x_j = x \cdot A \cdot {}^t x$, where $x = (x_0, x_1, \dots, x_n)$ is coordinate row, ${}^t x$ is its transposed column version, $A = \left(a_{ij}\right)$ is *symmetric* matrix with entries in $\Bbbk$. A is called *the Gram matrix* of q. Non-diagonal element $a_{ij} = a_{ji}$ of A equals one half of the coefficient at $x_i x_j$ in q.

In coordinate-free terms, for any quadratic polynomial q on V there exists a unique bilinear form $\tilde{q}(u, w)$ on $V \times V$ such that $q(x) = \tilde{q}(x, x)$. This form is called *the polarization* of q and is expressed in terms of q as $\tilde{q}(x, y) = \frac{1}{2}\big(q(x + y) - q(x) - q(y)\big) = \frac{1}{4}\big(q(x + y) - q(x - y)\big)$.

In coordinates, $\tilde{q}(x, y) = \sum a_{ij} x_i y_j = x \cdot A \cdot {}^t y = \frac{1}{2} \sum y_i \frac{\partial q(x)}{\partial x_i}$.

The polarization $\tilde{q}$ can be treated as a kind of scalar product[1] on V. Then the elements of the Gram matrix become the scalar products of basic vectors: $a_{ij} = \tilde{q}(e_i, e_j)$. In matrix notations $A = {}^t e \cdot e$, where $e = (e_0, e_1, \dots, e_n)$ is the row of basic vectors, ${}^t e$ is its transposed column version, and for $u, w \in V$ we put $u \cdot w \overset{\text{def}}{=} \tilde{q}(u, w)$. When we pass to another basis $e' = e \cdot C$, we change the Gram matrix A by $A' = {}^t C \cdot A \cdot C$. Under this change the determinant of the Gram matrix is multiplied by non zero square $\det^2 C \in \Bbbk^*$. Thus, modulo multiplication by non zero squares, *the Gram determinant* $\det q \overset{\text{def}}{=} \det A \in \Bbbk / \Bbbk^{*2}$ does not depend on a choice of basis. Quadric form q and quadric $Q = V(q)$ are called *smooth* (or *non-singular*), if $\det q \neq 0$.

Proposition 2.1 (Lagrange's theorem)
For any quadratic form q there exists a basis where the Gram matrix of q is diagonal.

Proof. Induction on $\dim V$. If $q \equiv 0$ or $\dim V = 1$, then the Gram matrix is diagonal. If $\dim V \geqslant 2$ and there is some $e \in V$ such that $q(e) = \tilde{q}(e, e) \neq 0$, we take $e_1 = e$ as the first vector of the desired basis. Each vector $v \in V$ has a unique decomposition $v = \lambda e + u$, where $\lambda \in \Bbbk$ and $u \in v^\perp = \{\, w \in V \mid \tilde{q}(v, w) = 0 \,\}$. Indeed, the orthogonality of v and $v - \lambda e$ forces $\lambda = \tilde{q}(e, v)/\tilde{q}(e, e)$, then it forces $u = v - (\tilde{q}(e, v)/\tilde{q}(e, e)) \cdot e$, and it does work actually.

Exercise 2.1. Verify that $v - (\tilde{q}(e, v)/\tilde{q}(e, e)) \cdot e \in e^\perp$.

Thus, $V = \Bbbk \cdot e \oplus e^\perp$. By induction, there is a basis $e_2, \dots, e_n \in e^\perp$ where the Gram matrix of the restricted form $q|_{e^\perp}$ is diagonal. Then $e_1, e_2, \dots, e_n$ is what we need. □

Corollary 2.1
If $\Bbbk$ is algebraically closed, then for any quadratic form q there exists a coordinate system such that $q(x) = x_0^2 + x_1^2 + \dots + x_k^2$, where $k \leqslant \dim V$.

Proof. Pass to a basis $\{e_i\}$ where the Gram matrix of q is diagonal, then divide all e_i such that $q(e_i) \neq 0$ by $\sqrt{q(e_i)}$. □

[1] possibly degenerated

Example 2.1 (quadrics on $\mathbb{P}_1$)
By Lagrange's theorem an arbitrary quadric on $\mathbb{P}_1$ has in appropriate homogeneous coordinates either an equation $x_0^2 = 0$ or an equation $x_0^2 + ax_1^2 = 0$, where $a \neq 0$.

The first quadric is singular ($\det q = 0$). It is called a *double point*, because its equation is squared linear equation of point $(0 : 1)$.

The second quadric $x_0^2 + ax_1^2 = 0$ is smooth. If $-a$ is not a square in $\Bbbk$ it is empty, certainly. If $-a = \delta^2$ is a square, then $x_0^2 + ax_1^2 = (x_0 - \delta x_1)(x_0 + \delta x_1)$ has two distinct roots $(\pm\delta : 1) \in \mathbb{P}_1$.

Since $-a \equiv -\det(q)$ up to multiplication by non zero squares , geometric type of an arbitrary quadric $V\left(a_0x_0^2 + 2a_1x_0x_1 + a_2x_1^2\right) \subset \mathbb{P}_1$ is completely predicted by its *discriminant*

$$D/4 \stackrel{\text{def}}{=} -\det(q) = a_1^2 - a_0a_2 \quad \text{(up to non-zero squares).}$$

If $D = 0$, then Q is a double point. If D is non-zero square, then Q is a pair of distinct points. If D is not a square, then $Q = \varnothing$, and this case never appears as soon $\Bbbk$ is algebraically closed.

2.2 Smoothness and singularities. It follows from the above example that a quadric Q can have precisely four different positional relationships with a line ℓ : either $\ell \subset Q$, or $\ell \cap Q$ consists of 2 distinct points, or $\ell \cap Q$ is a double point, or $\ell \cap Q = \varnothing$, and the latter case is never realised over an algebraically closed field.

Line ℓ is called *tangent* to quadric Q at point $p \in Q$, if either $p \in \ell \subset Q$ or $Q \cap \ell$ is the double point p. In these cases we say that ℓ *touches* Q at p. An union of all tangent lines touching Q at a given point $p \in Q$ is called *a tangent space* to Q at p and is denoted by T_pQ.

Proposition 2.2
Line (ab) touches quadric $Q = V(q)$ at the point $a \in Q$ iff $\tilde{q}(a, b) = 0$.

Proof. The Gram matrix of the restriction of q onto subspace $\Bbbk a \oplus \Bbbk b$ in the basis a, b is

$$\begin{pmatrix} \tilde{q}(a,a) & \tilde{q}(a,b) \\ \tilde{q}(a,b) & \tilde{q}(b,b) \end{pmatrix}.$$

Since $\tilde{q}(a, a) = q(a) = 0$ by assumption, the Gram determinant $\det q|_\ell = \tilde{q}^2(a, b)$ vanishes iff $\tilde{q}(a, b) = 0$. □

Corollary 2.2
For any point $p \notin Q$ an equation $\tilde{q}(p, x) = 0$ in $x \in \mathbb{P}_n$ defines a hyperplane

$$\Pi = \Pi(p) \stackrel{\text{def}}{=} \{x \in \mathbb{P}_n \mid \tilde{q}(p, x) = 0\} \tag{2-1}$$

that intersects Q along the set of all points where Q is touched by the tangent lines coming from p, i.e. $\Pi \cap Q$ is *an apparent contour* of Q from viewpoint p.

Proof. Since $\tilde{q}(p, p) = q(p) \neq 0$, the equation $\tilde{q}(p, x) = 0$ is non-zero linear equation on x. Thus, it defines a hyperplane. Everything else follows from prop. 2.2. □

2.2.1 Correlations. Associated with quadratic form q on V is linear mapping $\hat{q} : V \to V^*$ called *the correlation* of q and sending vector $v \in V$ to a linear form $\hat{q}(v) : w \mapsto \tilde{q}(w, v)$.

Exercise 2.2. Check that the matrix of $\hat{q}$ written in dual bases $\{e_i\} \subset V$, $\{x_i\} \subset V^*$ coincides with the Gram matrix of q in the basis $\{e_i\}$.

Thus, the rank of the Gram matrix $\operatorname{rk} A = \dim V - \dim \ker \hat{q}$ does not depend on a choice of basis. It is called *the rank* of quadratic form q and quadric Q. The space

$$\ker(q) \stackrel{\text{def}}{=} \ker \hat{q} = \{ v \in V \mid \tilde{q}(w, v) = 0 \ \ \forall\, w \in V \}$$

is called *the kernel* of q. Its projectivization

$$\operatorname{Sing} Q \stackrel{\text{def}}{=} \mathbb{P}(\ker q) = \{p \in \mathbb{P}(V) \mid \forall\, u \in V \ \hat{q}(p, u) = 0\}$$

is called *a vertex space* (or *a singular locus*) of quadric $Q \subset \mathbb{P}_n$. The points of $\operatorname{Sing} Q$ are called *singular*. The points of the complement $Q \smallsetminus \operatorname{Sing} Q$ are called *smooth*.

Corollary 2.3
Let $p \in Q \subset \mathbb{P}(V)$. Then the following conditions are equivalent:

$$1)\ p \in \operatorname{Sing} Q \qquad 2)\ T_p Q = \mathbb{P}(V) \text{ is the whole space} \qquad 3)\ \forall\, i \ \frac{\partial q}{\partial x_i}(p) = 0\,.$$

Corollary 2.4
If $p \in Q$ is smooth, then the tangent space $T_p Q = \{x \in \mathbb{P}_n \mid \tilde{q}(p, x) = 0\}$ is a hyperplane of codimension one in $\mathbb{P}_n$.

Theorem 2.1
Let $Q \subset \mathbb{P}(V)$ be an arbitrary quadric and $L \subset \mathbb{P}(V)$ be any projective subspace complementary to $\operatorname{Sing} Q$. Then $Q' = L \cap Q$ is non singular quadric in L and Q is a linear join[1] of Q' and $\operatorname{Sing} Q$.

Proof. Let $L = \mathbb{P}(U)$, that is $V = \ker q \oplus U$. Assume that there is $u \in U$ such that $\tilde{q}(u, u') = 0$ for all $u' \in U$. Since $\forall\, w \in \ker q \ \tilde{q}(u, w) = 0$ as well, we conclude that $\tilde{q}(u, v) = 0$ for all $v \in V$. This implies $u = 0$, because $\ker q \cap U = 0$. Hence, the restriction $q|_U$ is non singular.

If a line ℓ passes through a point $p \in \operatorname{Sing} Q$ and is not contained in Q, then $Q \cap \ell$ is the double point p. Thus, each line that intersects $\operatorname{Sing} Q$ either lies on Q or does not meet Q elsewhere. This means that each point $x \in Q \smallsetminus \operatorname{Sing} Q$ lies on Q together with all lines (xy), where y runs through $\operatorname{Sing} Q$. Since L is complementary to $\operatorname{Sing} Q$, all these lines intersect L, i.e. pass through Q'. □

2.2.2 Duality. Projective spaces $\mathbb{P}_n = \mathbb{P}(V)$ and $\mathbb{P}_n^\times \stackrel{\text{def}}{=} \mathbb{P}(V^*)$ associated with dual vector spaces V and V^* are called *dual* to each other. Geometrically, $\mathbb{P}_n^\times$ is the space of hyperplanes in $\mathbb{P}_n$ and vice versa. Indeed, linear equation $\langle\, \xi\,,\, v \,\rangle = 0$, being considered as an equation in $v \in V$ for a fixed $\xi \in V^*$, defines a hyperplane $\mathbb{P}(\operatorname{Ann} \xi) \subset \mathbb{P}_n$. On the other hand, as an equation on ξ for a fixed v, it defines a hyperplane in $\mathbb{P}_n^\times$ that consists of hyperplanes in $\mathbb{P}_n$ passing through v.

More generally, taking projectivizations in exrs. 1.9, we get for each $k = 0, 1, \dots, \dots n$ a bijection between k-dimensional projective subspaces $L \subset \mathbb{P}_n$ and $(n - k - 1)$-dimensional subspaces $H = \operatorname{Ann} L \subset \mathbb{P}_n^\times$. This bijection is involutive: $\operatorname{Ann} \operatorname{Ann} L = L$ and reverses the inclusions:

$$L \subset H \iff \operatorname{Ann} L \supset \operatorname{Ann} H\,.$$

[1] Given two sets X, Y in either $\mathbb{P}_n$ or $\mathbb{A}^n$, their *linear join* is the union of all lines (xy), where $x \in X$ and $y \in Y$

Geometrically, $\operatorname{Ann} L \subset \mathbb{P}_n^\times$ consists of all hyperplanes in $\mathbb{P}_n$ containing L.

The *projective duality* just described allows to translate true statements established in $\mathbb{P}_n$ to the dual statements about the dual figures in $\mathbb{P}_n^\times$ that have to be true as well but may look quite different from the originals. For example, a colinearity of 3 given points in $\mathbb{P}_n$ is translated as an existence of some codimension 2 subspace shared by 3 given hyperplanes in $\mathbb{P}_n^\times$.

2.2.3 Polar mappings. Quadratic form q and quadric Q are smooth iff the correlation mapping $\hat{q} : V \to V^*$ is an isomorphism. In this case it produces a linear projective isomorphism

$$\overline{q} : \mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}(V^*)$$

called *the polar mapping* (or just *the polarity*) of quadric $Q = V(q)$. The polarity sends point $p \in \mathbb{P}_n$ to hyperplane (2-1):

$$\Pi = \Pi(p) = \operatorname{Ann} \overline{q}(p) = \{x \in \mathbb{P}(V) \mid \tilde{q}(p, x) = 0\} ,$$

which cuts apparent contour of Q from viewpoint p in accordance with cor. 2.2. If $p \in Q$, then $\Pi = T_p Q$ is the tangent plane to Q at p. Hyperplane Π is called *a polar* of the point p and p is called *a pole* of Π w.r.t. Q.

Since the condition $\tilde{q}(a, b) = 0$ is symmetric in a and b, a point a lies on the polar of a point b iff b lies on the polar of a. Such the points are called *conjugated* w.r.t. quadric Q.

Proposition 2.3
Let a line (ab) intersect a smooth quadric Q in two distinct points c, d different from a, b. Then a, b are conjugated w.r.t. Q iff they are harmonic to c, d.

Proof. Chose some homogeneous coordinate $x = (x_0 : x_1)$ on the line $\ell = (ab) = (cd)$. The intersection $Q \cap \ell = \{c, d\}$ considered as a quadric in ℓ is given by quadratic form

$$q(x) = \det(x, c) \cdot \det(x, d) ,$$

whose polarization is $\tilde{q}(x, y) = \frac{1}{2}\big(\det(x, c) \cdot \det(y, d) + \det(y, c) \cdot \det(x, d)\big)$. Thus $\tilde{q}(a, b) = 0$ means that $\det(a, c) \cdot \det(b, d) = -\det(b, c) \cdot \det(a, d)$, i.e. $[a, b, c, d] = -1$. □

Proposition 2.4
Let $G, Q \subset \mathbb{P}_n$ be two quadrics with Gram matrices A and Γ at some basis of $\mathbb{P}_n$. If G is smooth, then the polar mapping of G sends Q to a quadric $Q_G^\times \subset \mathbb{P}_n^\times$ that has the same rank $\operatorname{rk} Q_G^\times = \operatorname{rk} Q$ and the Gram matrix $A^\times = \Gamma^{-1} A \Gamma^{-1}$ in the dual basis of $\mathbb{P}_n^\times$.

Proof. Let us write the homogeneous coordinates in $\mathbb{P}_n$ as row-vectors x and dual coordinates in $\mathbb{P}_n^\times$ as column vectors ξ. Then the polarity $\overline{g} : \mathbb{P}_n \xrightarrow{\sim} \mathbb{P}_n^\times$ associated with $G \subset \mathbb{P}_n$ sends $x \in \mathbb{P}_n$ to $\xi = \Gamma \cdot {}^t x$. Since Γ is invertible, we can recover x from ξ as $x = {}^t\xi \cdot \Gamma^{-1}$. When x runs through the quadric $xA{}^tx = 0$ the corresponding ξ runs through a quadric ${}^t\xi \cdot \Gamma^{-1} A \Gamma^{-1} \cdot \xi = 0$. □

Corollary 2.5
Tangent spaces of a smooth quadric $Q \subset \mathbb{P}_n$ form a smooth quadric $Q^\times \subset \mathbb{P}_n^\times$. The Gram matrices of Q and $Q^\times$ written in dual bases of $\mathbb{P}_n$ and $\mathbb{P}_n^\times$ are inverse to each other.

Proof. Put $G = Q$ and $\Gamma = A$ in prop. 2.4. □

2.2.4 Polarities over non-closed fields. If $\Bbbk$ is not algebraically closed, then there are non-singular quadratic forms q on V that produce empty quadrics $Q = V(q) = \varnothing$. However, their polarities $\overline{q} : \mathbb{P}(V) \to \mathbb{P}(V^*)$, considered as bijective correspondences between points and hyperplanes, are visible anyway.

Exercise 2.3. Describe geometrically the polarity defined by «imaginary» circle given by equation $x^2 + y^2 = -1$.

Thus, polarieties are more informative than quadrics, the more so since a quadric is recovered from its polarity as the set of all points lying on their polars.

Proposition 2.5
Two polarities coincide iff the corresponding quadratic forms are proportional.

Proof. This follows from lemma 1.2. □

Corollary 2.6
Over algebraically closed field two quadrics coincide iff their equations are proportional.

Proof. Let $Q = Q'$. Then $\operatorname{Sing} Q = \operatorname{Sing} Q'$. Hence, $\ker q = \ker q'$. Since the equations $q = 0$ and $q' = 0$ are not changed under the projection from the singular locus onto any complementary subspace $L \subset \mathbb{P}_n$ we can suppose that both quadrics are non singular. Over algebraically closed field we can take $n+2$ linearly generic points on a smooth quadric $Q = Q'$. They define the polar mapping uniquely. Thus, the correlations of the quadrics are proportional. □

2.2.5 Space of quadrics. All the polarities on $\mathbb{P}_n = \mathbb{P}(V)$ are in bijection with the points of the projective space $\mathbb{P}(S^2V^*) = \mathbb{P}_{\frac{n(n+3)}{2}}$. By this reason we call it *the space of quadrics*. For a given a point $p \in \mathbb{P}(V)$ the condition $q(p) = 0$ is linear in $q \in \mathbb{P}(S^2V^*)$. Hence, all quadrics passing through a given point form a projective hyperplane in the space of quadrics. Since any $n(n+3)/2$ hyperplanes in $\mathbb{P}_{n(n+3)/2}$ have non empty intersection, we come to the following conclusion

Proposition 2.6
Any collection of $n(n+3)/2$ points in $\mathbb{P}_n$ lies on some quadric. □

2.3 Conics. A quadric on a plane is called *a conic*. The space of conics in $\mathbb{P}_2 = \mathbb{P}(V)$ has dimension 5: $\mathbb{P}(S^2V^*) = \mathbb{P}_5$. Over algebraically closed field there 3 geometrically different conics:

double line has rank 1 and is given in appropriate coordinates by equation $x_0^2 = 0$; it is totally singular, i.e. has no smooth points at all;

split conic (or a pair of crossing lines) has rank 2 and is given in appropriate coordinates by equation $x_0^2 + x_1^2 = 0$; this is a union of two lines $x_0 = \pm\sqrt{-1} \cdot x_1$ crossing at the singular point[1] $(0 : 0 : 1)$

smooth conic has rank 3 and is given in appropriate coordinates by equation $x_0^2 + x_1^2 + x_2^2 = 0$.

[1]note that this agrees with theorem 2.1

2.3.1 Rational parametrization. Over any field $\Bbbk$ each non-empty smooth conic $C \subset \mathbb{P}_2$ admits *a rational parametrization* by quadratic polynomials, that is a mapping $\varphi : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_2$ given in homogeneous coordinates by a coprime triple of homogeneous polynomials of degree 2

$$\varphi(t_0, t_1) = \big(\varphi_0(t_0, t_1) : \varphi_1(t_0, t_1) : \varphi_2(t_0, t_1)\big) \in \mathbb{P}_2 \tag{2-2}$$

and providing a bijection between $\mathbb{P}_1$ and the conic $C \subset \mathbb{P}_2$. Such a parametrization is provided by a projection $p : C \xrightarrow{\sim} \ell$ from any point $p \in C$ to any line $\ell \not\ni p$.

Indeed, each line (pt) that joins p with some $t \in \ell$ and differs from the tangent line T_pC at p does cross C in two distinct points, one of which is p. The second point $t + \lambda p$ is defined from the equation $\tilde{q}(t + \lambda p, t + \lambda p) = 0$, which is reduced to $q(t) = -2\lambda\tilde{q}(t, p)$. Thus, a desired parametrization $\ell \xrightarrow{\sim} C$ takes $t \in \ell$ to $\varphi(t) = q(t) \cdot p - 2q(p, t) \cdot t \in C$.

Exercise 2.4. Make sure that coordinates of $\varphi(t) \in \mathbb{P}_2$ are homogeneous quadratic polynomials in $t \in \ell$ and verify that $\varphi(t)$ is well defined at $t = T_pC \cap \ell$ and sends it to $p \in C$.

Over algebraically closed field $\Bbbk$ any smooth conic can be identified by appropriate choice of coordinates with *the Veronese conic*[1] C_{ver} given by determinant equation

$$\det \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix} = a_0 a_2 - a_1^2 = 0 \tag{2-3}$$

and coming with built-in rational parametrization (2-3) via

$$(\alpha_0 : \alpha_1) \mapsto (a_0 : a_1 : a_2) = (\alpha_0^2 : \alpha_0\alpha_1 : \alpha_1^2)\,. \tag{2-4}$$

This allows to parametrize any smooth conic as well.

Exercise 2.5. Verify that projection of the Veronese conic $a_0a_2 - a_1^2 = 0$ from point $(1 : 1 : 1)$ to line $a_1 = 0$ takes $(\alpha_0^2 : \alpha_0\alpha_1 : \alpha_1^2) \mapsto (\alpha_0 : 0 : \alpha_1)$.

Proposition 2.7
For any plane curve F of degree d and any smooth conic C the intersection $C \cap F$ either consists of at most $2d$ points or coincides with C.

Proof. Let $F \subset \mathbb{P}_2$ be given by homogeneous equation $f(x) = 0$ of degree d and $C \subset \mathbb{P}_2$ be parametrized via some mapping $\varphi : \mathbb{P}_1 \to \mathbb{P}_2$ given by a triple of homogeneous quadratic polynomials (2-2). Then the values of t that produce the intersection points $C \cap F$ satisfy the equation $f\big(q(t)\big) = 0$, whose L.H.S. either vanishes identically or is a non-zero homogeneous polynomial of degree $2d$. In the first case $C \subset F$. In the second case there are at most $2d$ solutions. □

Corollary 2.7
Any 5 points in $\mathbb{P}_2$ lie on some conic C. It is unique iff no 4 of the points are collinear. If no 3 of the points are collinear, then C is smooth.

[1]recall from example 1.9 on p. 17 that $C_{\text{ver}} \subset \mathbb{P}_2 = \mathbb{P}(S^2U^*)$ consists of all quadratic forms $a_0\,x_0^2 + 2a_1\,x_0x_1 + a_2\,x_1^2$ on the space $U = \Bbbk^2$ with coordinates (x_0, x_1) that are perfect squares of linear forms $\alpha_0x_0 + \alpha_1x_1 \in U^*$

Proof. The first statement follows from prop. 2.6 for $n = 2$. If some 3 of given points lie in the same line ℓ, then ℓ is a component of any conic C passing through given points. If no one of two remaining given points a, b does lie on ℓ, then $C = \ell \cup (ab)$ is unique. If some one does, say $a \in \ell$, then any split conic $\ell \cup \ell'$, where $\ell' \ni a$, passes through all 5 given points. If no 3 of 5 given points are collinear, then any conic C passing through them has to be smooth, because otherwise C consists of either 1 or 2 lines and one of them should contain at least 3 of given points. By prop. 2.7, any two smooth conics intersecting in 5 distinct points coincide. □

Corollary 2.8
In $\mathbb{P}_2$ any 5 five lines without triple intersection do touch a unique smooth conic.

Proof. This is projectively dual to cor. 2.7. □

Example 2.2 (homographies via conics)
Let a homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ between two lines $\ell_1, \ell_2 \subset \mathbb{P}_2$ send an ordered triple of distinct points $a_1, b_1, c_1 \in \ell_1$ distinct from the intersection $q = \ell_1 \cap \ell_2$ to an ordered triple $a_2, b_2, c_2 \in \ell_2$. Geometrically, there are two ways to do that (see fig. 2⋄1 and fig. 2⋄2): either the lines (a_1a_2), (b_1b_2), (c_1c_2) meet all together at some point p or there are no triple intersections among 5 lines ℓ_1, ℓ_2, (a_1, a_2), (b_1, b_2), (c_1, c_2).

In the first case, by example 1.12, homography φ is the perspective with the centre at p. This case is characterized by the property $\varphi(q) = q$.

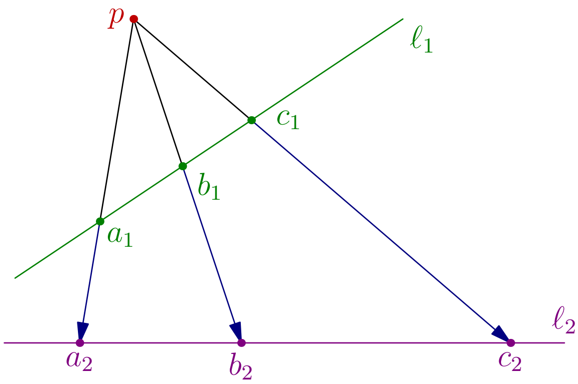

Рис. 2⋄1. Perspective $p : \ell_1 \to \ell_2$.

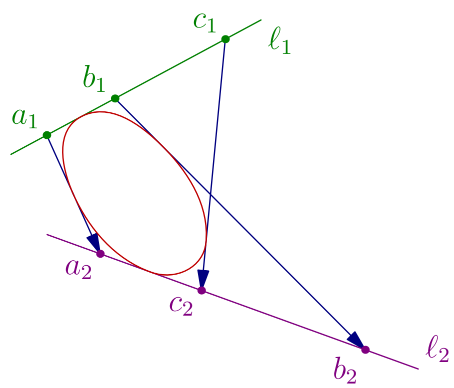

Рис. 2⋄2. Homography $C : \ell_1 \to \ell_2$.

In the second case, by cor. 2.8, there is a unique smooth conic C touching all 5 lines ℓ_1, ℓ_2, (a_1, a_2), (b_1, b_2), (c_1, c_2). We claim that φ is cut by its tangent lines, that is, sends $x \in \ell_1$ to the intersection point of ℓ_2 with tangent line to C coming from x and different from ℓ_1.

Indeed, the map $C : \ell_1 \to \ell_2$ just described is clearly bijective. It is also rational: equations for tangent lines from x to C are the intersections of dual conic $C^\times \subset \mathbb{P}_2^\times$ with line $\operatorname{Ann} x \subset \mathbb{P}_2$, i.e. they are the roots of quadratic equation; since on of them, namely the equation of ℓ_1, is known, the second is a rational function in coordinates of x and coefficients of equations for C and ℓ_1. Hence, $C : \ell_1 \to \ell_2$ is a homography by lemma 1.3. Since its action on a_1, b_1, c_1 agrees with φ, they coincide. Note that the image and the pre-image of the intersection point $q = \ell_1 \cap \ell_2$ are the touch points $\ell_2 \cap C$ and $\ell_1 \cap C$ respectively.

Thus, any homography $\varphi : \ell_1 \xrightarrow{\sim} \ell_2$ either is a perspective or is cut by tangents to a smooth conic tangent to both lines ℓ_1, ℓ_2. Anyway, the centre p of the perspective and the conic C are *uniquely* predicted by φ. A perspective can be considered as a degeneration of a non-perspective homography $C : \ell_1 \to \ell_2$ arising when C becomes split in two lines crossing at the centre of the perspective[1].

Proposition 2.8 (inscribed circumscribed triangles)
Two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are both inscribed in some smooth conic Q' iff they are both circumscribed about some smooth conic Q''.

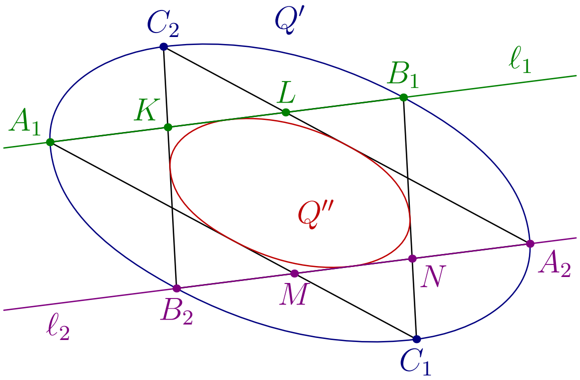

Рис. 2◇3. Inscribed circumscribed triangles.

Proof. Let 6 points $A_1, B_1, C_1, A_2, B_2, C_2$ lie on a smooth conic Q' like in fig. 2◇3. Put $\ell_1 = (A_1B_1)$, $\ell_2 = (A_2B_2)$ and write $C_2 : \ell_1 \simeq Q'$ for projection of ℓ_1 onto Q' from C_1 and $C_1 : Q' \simeq \ell_2$ for projection of Q' onto ℓ_2 from C_2. Their composition

$$[C_1 : Q' \simeq \ell_2] \circ [C_2 : \ell_1 \simeq Q'] \; : \; \ell_1 \simeq \ell_2$$

is a non-perspective homography sending $A_1 \mapsto M$, $K \mapsto B_2$, $L \mapsto A_2$, $B_1 \mapsto N$. Let Q'' be a smooth conic whose tangent lines cut this homography. Then Q'' is obviously inscribed in the both triangles. The opposite implication is dual to just proven. □

Corollary 2.9 (Poncelet's porism for triangles)
Assume that there exist a triangle $A_1B_1C_1$ simultaneously inscribed in some smooth conic Q' and circumscribed about some smooth conic Q''. Then each point of Q' except for a finite set is a vertex of a triangle inscribed in Q' and simultaneously circumscribed about Q''.

Proof. Pick up any $A_2 \in Q$ and find $B_2, C_2 \in Q'$ such that lines (A_2B_2) and (A_2C_2) touch Q'' (see fig. 2◇3). By prop. 2.8, triangles $A_1B_1C_1$ and $A_2B_2C_2$ are both circumscribed about some smooth conic that does touch 5 lines (AB), (BC), (CA), (A_2B_2), (A_2C_2). Since there is exactly one such conic, it coincides with Q''. □

Example 2.3 (Tracing conic by ruler)
There is an effective method for constructing the points of a smooth conic passing through given 5 points $p_1, p_2, \dots, p_5$ by means of a ruler only. First of all draw the lines $\ell_1 = (p_2p_5)$, $\ell_2 = (p_2p_4)$ and pick up the points $p = (p_1p_4) \cap (p_3p_5)$ and $q = \ell_1 \cap (p_2p_4)$ like on. fig. 2◇4. We claim that

[1]these lines can be chosen in many ways: any two lines joining the corresponding points are fitted in the picture

perspective $p : \ell_1 \rightsquigarrow \ell_2$ coincides with projection $p_1 : \ell_1 \rightsquigarrow C$ of ℓ_1 onto C from p_1 followed by projection $p_3 : C \rightsquigarrow \ell_2$ from C onto ℓ_2 from $p_3 \in C$, because their actions on $p_2, p_5, q \in \ell_1$ coincide. Thus, if we draw any line ℓ through p and pick up the intersections $x_1 = \ell \cap \ell_1$ and $x_2 = \ell \cap \ell_2$, then a point $c(\ell) = (p_1 x_1) \cap (p_2 x_2)$ lies at C (see fig. 2⋄4) and runs through the whole of C as ℓ is rotated about p. This construction can be reformulated es follows.

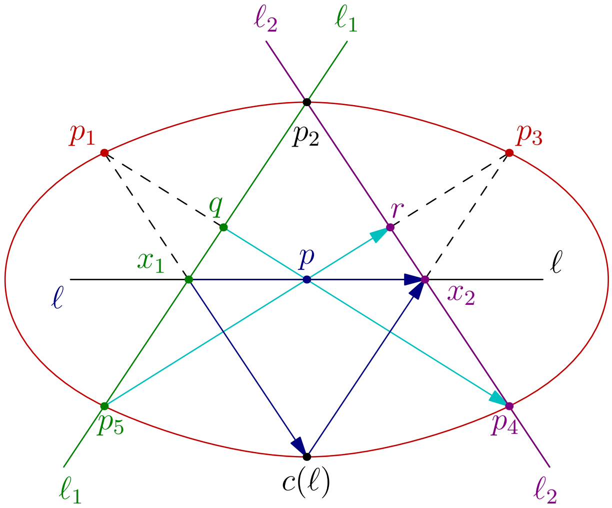

Рис. 2⋄4. Tracing a conic by a ruler.

Theorem 2.2 (Pascal's theorem)
Six points $p_1, p_2, \dots, p_6$ no 3 of which are collinear do lie on a smooth conic iff 3 intersection points[1] $x = (p_3 p_4) \cap (p_6 p_1)$, $y = (p_1 p_2) \cap (p_4 p_5)$, $z = (p_2 p_3) \cap (p_5 p_6)$ are collinear (see fig. 2⋄6).

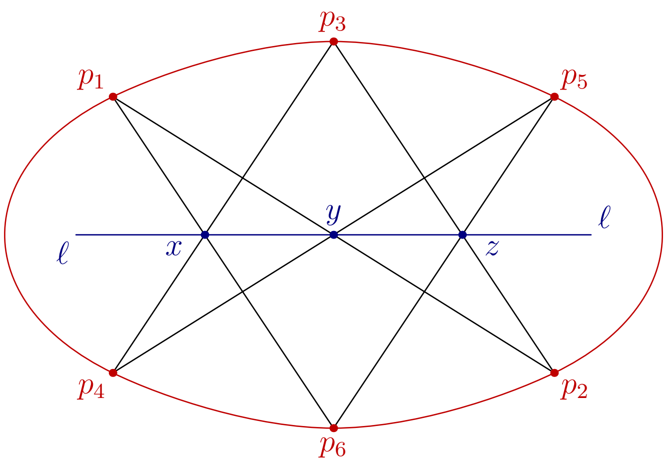

Рис. 2⋄5. The hexogram of Pascal.

Proof. Let $\ell_1 = (p_3 p_4)$, $\ell_2 = (p_3 p_2)$ as in fig. 2⋄5. Since $z \in (xy)$, the perspective

$$y : \ell_1 \to \ell_2 , \tag{2-5}$$

[1]they can be viewed as intersections of «opposite sides» of hexagon $p_1, p_2, \dots, p_6$

sends $x \mapsto z$. As we have seen in example 2.3 , this perspective is composed from projections

$$\left(p_5 : C \xrightarrow{\sim} \ell_2\right) \circ \left(p_1 : \ell_1 \xrightarrow{\sim} C\right) \tag{2-6}$$

where C is the smooth conic passing trough $p_1, p_2, \dots, p_5$. Thus, $p_6 = (p_5 z) \cap (p_3 x) \in C$. Vice versa, if $(p_5 z) \cap (p_3 x) \in C$, then $x \in \ell_1$ goes to $z \in \ell_2$ under the composition (2-6). Hence, perspective (2-5) also takes $x \mapsto z$ forcing $z \in (xy)$. $\square$

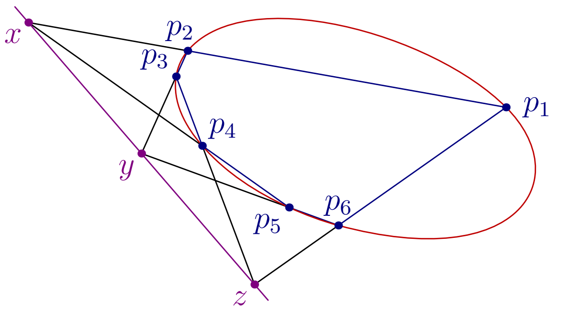

Рис. 2◇6. Inscribed hexagon.

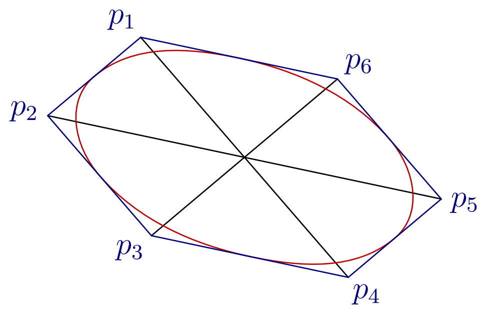

Рис. 2◇7. Circumscribed hexagon.

Corollary 2.10 (Brianchon's theorem)
A hexagon $p_1, p_2, \dots, p_6$ is circumscribed around a non singular conic iff its main diagonals $(p_1 p_4)$, $(p_2 p_5)$, $(p_3 p_6)$ are intersecting at one point (see fig. 2◇7).

Proof. This is dual to the Pascal's theorem 2.2. $\square$

2.3.2 Internal geometry of a smooth conic. During this section we assume that the ground field $\Bbbk$ is algebraically closed (and $\mathrm{char}(\Bbbk) \neq 2$). Any projection $p : C \to \ell$ from a point $p \in C$ to a line $\ell \ni p$ establishes birational bijection $C \simeq \mathbb{P}_1$ and any two such bijections, coming under different choices of p and ℓ, differ from each other by a linear fractional automorphism of $\mathbb{P}_1$, because if we chose another $p' \in C$ and another line ℓ', then the composition

$$\left(p' : C \xrightarrow{\sim} \ell'\right) \circ \left(p : \ell \xrightarrow{\sim} C\right)$$

provides a homography $\ell \xrightarrow{\sim} \ell'$. This allows to introduce internal homogeneous coordinates $(\vartheta_0 : \vartheta_1)$ on C defined up to linear automorphism: we choose some homogeneous coordinates in ℓ and lift them to C via projection $p : C \to \ell$.

As soon some internal homogeneous coordinates in C were fixed up a linear automorphism, we can define the cross-ratio for a quadruple of distinct points $c_i \in C$ in terms of these coordinates by the usual formula

$$[c_1, c_2, c_3, c_4] \overset{\text{def}}{=} \frac{\det\left(c_1, c_3\right) \cdot \det\left(c_2, c_4\right)}{\det\left(c_1, c_4\right) \cdot \det\left(c_2, c_3\right)} . \tag{2-7}$$

Being invariant under homographies, it does not depend on a choice of internal homogeneous coordinates. Geometrically, (2-7) is nothing but the cross-ratio of 4 lines $\ell_i = (cc_i)$, which join c_i with some fifth point $c \in C$, in the pencil of lines passing through p.

Exercise 2.6. Show that the latter description agrees with the previous one and does not depend on a choice of the 5 th point $c \in C \smallsetminus \{c_1, c_2, c_3, c_4\}$.

We say that a mapping $\varphi : C \to C$ is *a homography* on C, if it can be treated as invertible linear transformation of internal homogeneous coordinates[1]. Of course, any ordered triple of distinct points in C is sent to any other such a triple by a unique homography and two quadruples of distinct points can be sent to each other by some homography iff their cross-ratios coincide.

Example 2.4 (involutions)
Non-identical homography inverse to itself is called *involution*. Let an involution $\sigma : C \to C$ interchange $a' \leftrightarrow a''$ and $b' \leftrightarrow b''$ like on fig. 2⋄8 and let $s = (a'a'') \cap (b'b'') \in \mathbb{P}_2 \smallsetminus C$. We claim that σ is cut by the pencil of lines passing through s, i.e. $\sigma : c' \leftrightarrow c''$ iff $(c'c'') \ni s$.

Indeed, the latter rule provides an involution on C, because it is bijective and birational[2]. Since it action on 4 points a', a'', b', b'' agrees with σ, it coincides with σ everywhere. Point s is called *the centre* of involution σ.

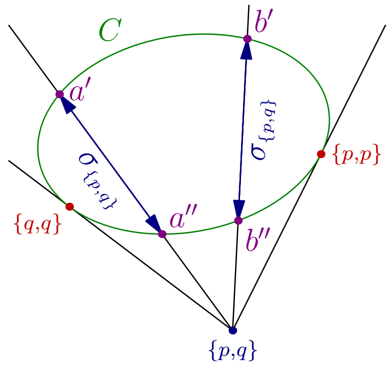

Fig. 2⋄8. Involution of onic.

We conclude immediately that each involution on $\mathbb{P}_1$ has exactly two fixed points and for any pair of distinct points there exists a unique involution keeping these two points fixed. Indeed, fixed points of σ are the intersection points of C with the polar line of s w.r.t. C.

Actually, as soon we are given with a smooth conic $C \subset \mathbb{P}_2$, we can canonically identify the points of $\mathbb{P}_2$ with non-ordered pairs $\{r, s\}$ of points $r, s \in C$ by sending $\{r, s\} \subset C$ to the intersection of tangents $T_rC \cap T_sC \in \mathbb{P}_2$ when $r \neq s$ and interpreting the double points $\{p, p\}$ as the pints p of C itself. Under this identification, an involution σ with the centre $s = \{p, q\}$ has $p, q \in C$ as its fixed points and takes $a \leftrightarrow b$ iff pair $\{a, b\}$ is harmonic to the pair of fixed points, i.e. iff $[a, b, p, q] = -1$.

Exercise 2.7. Verify all these statements.

Exercise 2.8. In the notations of prop. 2.3 on p. 30 consider an involution of line (ab) provided by conjugation w.r.t. Q (that is, $x \leftrightarrow y$ as soon x lies on the polar of y). Use the above statements to prove prop. 2.3 on p. 30 purely geometrically without any computations.

Example 2.5 (cross-axis for a homography on a conic)
It is evident that a homography $\varphi : C \to C$ taking a_1, b_1, c_1 to a_2, b_2, c_2 splits into composition of projection $b_2 : C \to \ell$ followed by projection $b_1 : \ell \to C$, where the line ℓ joins cross-intersection points $(a_1, b_2) \cap (b_1, a_2)$ and $(c_1, b_2) \cap (b_1, c_2)$, see fig. 2⋄9. Hence, the fixed points of φ are the points of intersection $\ell \cap C$. Thus, ℓ does not depend on a choice of an ordered triple $a_1, b_1, c_1 \in C$. We conclude also that φ has either two distinct fixed points (when ℓ does not touch C) or a unique fixed point (when ℓ touches C at this point).

Being independent on a_1, b_1, c_1, the line ℓ coincides with the locus of all cross-intersections $(x, \varphi(y)) \cap (y, \varphi(x))$ as $x \neq y$ independently run through C . Besides other beauties, this gives

[1] again, this does not depend on a choice of coordinates

[2] coordinates of two intersection points of a given conic with a given line are rational functions of each other rationally depending in coefficients of equations for the conic and the line (this is a version of the Vieta formula: the sum of two roots of a quadratic equation equals the ratio of appropriate coefficients of the equation)

another proof of Pascal's theorem: intersections of opposite sides of a hexagon $a_1c_2b_1a_2c_1b_2$ inscribed in C lie on the cross – axis of a homography that sends a_1, b_1, c_1 to a_2, b_2, c_2.

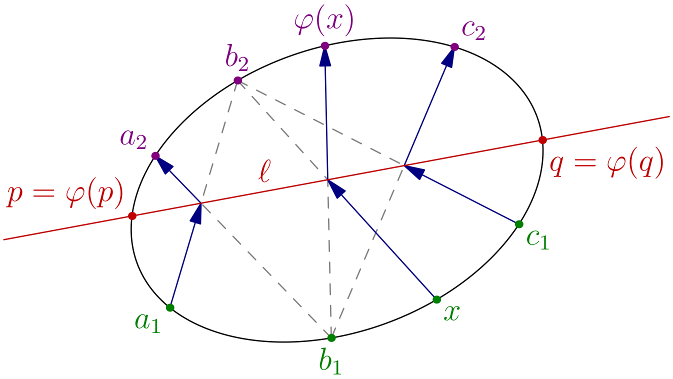

Рис. 2◇9. Cross-axis.

Using only a ruler, it is easy to draw the cross-axis of a homography $\varphi : C \to C$ as soon as its action on some triple of distinct points is given. This allows, using only a ruler, to construct an image of a given point under such a homography as well as to pick up its fixed points.

For example, let us draw a pair of tangent lines to a given smooth conic C coming to C from a given point $s \notin C$. The touch points are the fixed points of the involution on C with the centre at s. To find them, draw any 3 secant lines passing through s, then draw the cross-axis ℓ_{σ_s} (see fig. 2◇10) and pick up $\ell_{\sigma_s} \cap C$. In prb. 2.6 on p. 49 you will find even shorter drawing.

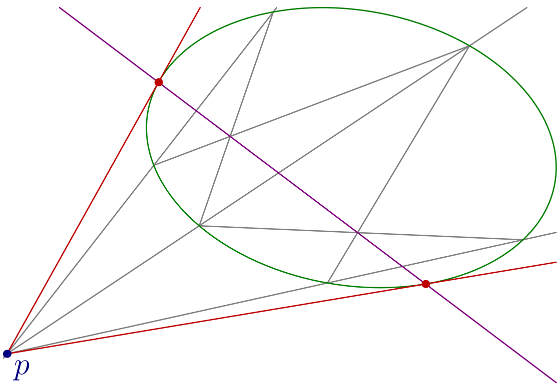

Рис. 2◇10. Drawing tangent lines.

2.4 Quadratic surfaces in $\mathbb{P}_3 = \mathbb{P}(V)$ form 9-dimensional projective space $\mathbb{P}_9 = \mathbb{P}(S^2V^*)$. Over algebraically closed field $\Bbbk$ there are 4 classes of projectively equivalent quadratic surfaces, distinguished by the rank:

double plane has rank 1 and can be given by equation $x_0^2 = 0$; it is totally singular

split quadric (or a pair of crossing planes) has rank 2 and can be given by equation $x_0^2 + x_1^2 = 0$; it is a union of two planes $x_0 = \pm\sqrt{-1} \cdot x_1$ crossing along the singular line[1] $x_0 = x_1 = 0$

[1]thus, the quadric is the linear join of this line with two points laying on some complementary line in agreement with theorem 2.1

quadratic cone $x_0^2 + x_1^2 + x_2^2 = 0$ has rank 3; in agreement with theorem 2.1 on p. 29, it is linear joint of the singular point $(0:0:0:1)$ with a smooth conic given by the same equation in the plane $x_3 = 0$ complementary to $(0:0:0:1)$

smooth quadratic surface has rank 4 and can be written as $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ but admits much more convenient determinantal model decribed below.

2.4.1 The Segre quadric. We fix two 2-dimensional vector spaces U_-, U_+, put

$$W = \mathrm{Hom}(U_-, U_+) = \{\text{linear maps } f : U_- \to U_+\},$$

and consider $\mathbb{P}_3 = \mathbb{P}(W)$. A choice of bases in $U_\pm$ identifies this space with the projectivization $\mathbb{P}\left(\mathrm{Mat}_{2\times 2}(\Bbbk)\right)$ of the space of 2×2-matrices

$$\begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix}.$$

The determinant $f \mapsto \det(f)$ is a smooth quadratic form on W. In the matrix notations it takes

$$\begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} \mapsto \alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}$$

Its zero set $Q_s \stackrel{\text{def}}{=} V(\det) = \{f \neq 0 \mid \det f = 0\}$ is called *the Segre quadric*. It consists of all operators of rank 1 considered up to proportionality.

Exercise 2.9. Show that any $m \times n$ matrix A of rank one is a product of some m-column and n-row, which are uniquely predicted by A up to proportionality.

Recall that any vector $v \in U_+$ and any covector $\xi \in U_-^*$ produce rank 1 operator

$$\xi \otimes v : U_- \to U_+, \quad u \mapsto \langle\, \xi\,, u\,\rangle \cdot v$$

and any rank 1 operator f has the form $f = \xi \otimes v$, where v spans $\mathrm{im}\, f$ and ξ spans $\mathrm{Ann}\ker f$ (i.e. ξ is a linear equation for a hyperplane $\ker f \subset U_-$). Thus, the Segre quadric coincides with an image of *the Segre embedding*

$$s : \mathbb{P}_1 \times \mathbb{P}_1 = \mathbb{P}(U_-^*) \times \mathbb{P}(U_+) \hookrightarrow \mathbb{P}(\mathrm{Hom}(U_-, U_+)) = \mathbb{P}_3 \tag{2-8}$$

that sends $(\xi, v) \in U_-^* \times U_+$ to rank 1 operator $\xi \otimes v \in W$. If we choose some coordinates $(\xi_0 : \xi_1)$ in U_-^* and $(t_0 : t_1)$ in U_+, the operator $\xi \otimes v$ gets the matrix

$$\xi \otimes v = \begin{pmatrix} \xi_0 t_0 & \xi_1 t_0 \\ \xi_0 t_1 & \xi_1 t_1 \end{pmatrix}.$$

So, the Segre embedding gives a rational parametrization

$$\alpha_{00} = \xi_0 t_0\,, \quad \alpha_{01} = \xi_1 t_0\,, \quad \alpha_{10} = \xi_0 t_1\,, \quad \alpha_{11} = \xi_1 t_1\,. \tag{2-9}$$

for the Segre quadric by pairs $\left((\xi_0 : \xi_1), (t_0 : t_1)\right) \in \mathbb{P}_1 \times \mathbb{P}_1$.

Note that $\mathbb{P}_1 \times \mathbb{P}_1$ is ruled by two families of «coordinate» lines $\xi \times \mathbb{P}_1$ and $\mathbb{P}_1 \times t$. Let us call them *the first* and *the second* ruling line families. Since the parametrization (2-9) takes lines to lines, we get

Proposition 2.9
The Segre embedding sends each coordinate line family to a ruling of Q_s by a family of pairwise skew lines. These two line families exhaust all the lines on Q_s. Any two lines from different families are intersecting and each point of Q_s is the intersection point of two lines from different families.

Proof. A line $\xi \times \mathbb{P}_1$, where $\xi = (\xi_0 : \xi_1) \in \mathbb{P}(U_-^*)$, goes to a set of all rank 1 matrices with

$$(\text{1-st column}) : (\text{2-nd column}) = \xi_0 : \xi_1 .$$

They form a line in $\mathbb{P}_3$ given by two linear equations $a_{00} : a_{01} = a_{10} : a_{11} = \xi_0 : \xi_1$. Analogously, $s(\mathbb{P}_1 \times t)$, where $t = (t_0 : t_1) \in \mathbb{P}(U_+)$, goes to the line $a_{00} : a_{10} = a_{01} : a_{11} = t_0 : t_1$, which consists of all rank 1 matrices with

$$(\text{1-st row}) : (\text{2-nd row}) = t_0 : t_1 .$$

Since the Segre mapping is bijective, each line family consists of pairwise skew lines, any two lines from the different families are intersecting, and for any $x \in Q_s$ there is a pair of lines intersecting at x and belonging to different families. This forces $Q_s \cap T_x Q_s$ to be a split conic and implies that there are no other lines on Q_s. □

Corollary 2.11
Any 3 lines on $\mathbb{P}_3$ lie on some quadric. If the lines are mutually skew, then this quadric is unique, non singular, and is ruled by all lines in $\mathbb{P}_3$ intersecting all 3 given lines.

Proof. Pick up a triple of distinct points on each line. By prop. 2.6 applied for $n = 3$, there exists a quadric passing through these 9 points. It contains all 3 lines. Since a singular quadric does not contain a triple of mutually skew lines, any quadric passing through 3 pairwise skew lines is smooth and ruled by two families of lines. Clearly, all 3 given lines lie in the same family. Then the second ruling family is nothing else as the set of lines in $\mathbb{P}_3$ intersecting all 3 given lines. Thus the quadric is unique. □

Exercise 2.10. How many lines intersect 4 given pairwise skew lines in $\mathbb{P}_3$?

2.5 Linear subspaces lying on a smooth quadric. Line rulings from prop. 2.9 have higher dimensional versions. We say that a smooth quadric Q is *k-planar*, if there is a projective subspace $L \subset Q$ of dimension $\dim L = k$ but Q does not contain projective subspaces of any bigger dimension. By the definition, the planarity of an empty quadric equals -1. Thus, quadrics of planarity 0 are non-empty and do not contain lines.

Proposition 2.10
The planarity of a smooth quadric $Q \subset \mathbb{P}_n$ over any field $\Bbbk$ is at most $[(n-1)/2]$, i.e. does not exceed $\dim Q/2$.

Proof. Let $\mathbb{P}_n = \mathbb{P}(V)$, $\dim V = n+1$, and $L = \mathbb{P}(W) \subset Q = V(q)$, where $q \in S^2V^*$. Since $q|W \equiv 0$, the correlation $\hat{q} : V \xrightarrow{\sim} V^*$ sends W inside $\operatorname{Ann}(W) \subset V^*$. Injectivity of $\hat{q}$ implies

$$\dim(W) = \dim \hat{q}(W) \leqslant \dim \operatorname{Ann} W = \dim V - \dim W .$$

Thus, $2\dim W \leqslant \dim V$ and $2\dim L \leqslant n-1$. □

Lemma 2.1
For any smooth quadric Q and hyperplane Π the intersection $\Pi \cap Q$ is either a smooth quadric in Π or has precisely one singular point $p \in \Pi \cap Q$. The latter takes place iff $\Pi = T_p Q$.

Proof. Let $Q = V(q) \subset \mathbb{P}(V)$ and $\Pi = \mathbb{P}(W)$. Since

$$\dim \ker \left(\hat{q}|_W\right) = \dim \left(W \cap \hat{q}^{-1}(\operatorname{Ann} W) \right) \leqslant$$
$$\leqslant \dim \hat{q}^{-1}(\operatorname{Ann} W) = \dim \operatorname{Ann} W = \dim V - \dim W = 1$$

quadric $\Pi \cap Q$ has at most one singular point. If $\operatorname{Sing} Q = \{p\} \neq \varnothing$, then p spans 1-dimensional space $\ker \hat{q}|_W \subset W$ and $\operatorname{Ann}(\hat{q}(p)) = W$. Hence, $T_p Q = \Pi$. Vice versa, if $\Pi = T_p Q = \mathbb{P}(\operatorname{Ann} \hat{q}(p))$, then $p \in \operatorname{Ann} \hat{q}(p)$ belongs to the kernel of the restriction of $\hat{q}$ onto $\operatorname{Ann} \hat{q}$. □

Proposition 2.11
Let $Q \subset \mathbb{P}_{n+1}$ be a smooth n-dimensional quadric over any field $\Bbbk$. Then m-dimensional projective subspaces $L \subset Q$ passing through a fixed point $p \in Q$ stay in bijection with $(m-1)$-dimensional projective subspaces L' laying on $(n-2)$-dimensional smooth quadric $Q' = H \cap Q$ cut out of Q by an arbitrary hyperplane[1] $H \subset T_p Q$ complementary to p.

Proof. Each m-dimensional projective subspace $L \subset Q$ passing through $p \in Q$ lies in $Q \cap T_Q$. By lemma 2.1, the latter intersection is a singular quadric with $\operatorname{Sing}(Q \cap T_Q) = p$. By theorem 2.1 on p. 29, $Q \cap T_Q$ is a simple cone with vertex p over a smooth quadric Q' cut out of Q by any hyperplane $H \subset T_p Q$ that does not pass through p. Hence $(m-1)$-dimensional subspace $L' = L \cap H = L \cap Q'$ lies on Q'. Vice versa, a linear join of p with an arbitrary $(m-1)$-dimensional projective subspace $L' \subset Q'$ containes p and lies on Q. □

Corollary 2.12
A cardinality of the set of k-dimensional projective subspaces laying on Q and passing through a given point $p \in Q$ does not depend on $p \in Q$. In particular, each point of a smooth m-planar quadric Q lies in some m-dimensional projective subspace laying on Q.

Proof. If $a, b \in Q$ and $b \notin T_a Q$, then $H = T_a Q \cap T_b Q$ does not pass through a, b and lies in the both tangent spaces $T_a Q$, $T_b Q$ as a hyperplane. By prop. 2.11, the set of k-dimensional projective subspaces of Q passing through a as well as the set of k-dimensional projective subspaces of Q passing through b both stay in bijection with the set of all $(k-1)$-dimensional projective subspaces of $Q \cap H$ в H. Thus, they are of the same cardinality. If $b \in T_a Q$, pick up any point $c \in Q \smallsetminus (T_a Q \cup T_b Q)$. Then the set of k-dimensional projective subspaces of Q passing through c stays in bijection with those sets through a and through b. □

Corollary 2.13
A smooth n-dimensional quadric over algebraically closed field is $[n/2]$-planar.

Proof. This holds for $n = 0, 1, 2$. Since all smooth n-dimensional quadrics over algebraically closed field are projectively equivalent to each other, the result follows from prop. 2.11 via induction in n. □

[1] note that $\dim H = n - 1$

2.6 Digression: orthogonal geometry over arbitrary field. Let V be a vector space over an arbitrary field of characteristic $\neq 2$ and $\beta : V \times V \to \Bbbk$ be non-degenerated symmetric bilinear form. Latter could be expressed by any of the following mutually equivalent properties:

- for any non zero $v \in V\ \exists\, v' \in V : \beta(v', v) \neq 0$
- for any covector $\psi : V \to \Bbbk$ there exists a unique $w_\psi \in V : \psi(v) = \beta(v, w_\psi)\ \forall\, v \in V$
- correlation $\widehat{\beta} : V \xrightarrow{\sim} V^*$, which sends $v \in V$ to linear form $\widehat{\beta}(v) : w \mapsto \beta(w, v)$, is an isomorphism
- the Gram matrix $B = \big(\beta(e_i, e_j)\big)$ is non-degenerated for some (thus, for any) basis $\{e_i\} \subset V$.

We call such β *a scalar product* on V.

Proposition 2.12
If a symmetric form[1] $\beta : V \times V \to \Bbbk$ induces non-degenerated scalar product on some subspace $U \subset V$, then $V = U \oplus U^\perp$, where $U^\perp = \{v \in V \mid \forall\, u \in U\ \beta(u, v) = 0\}$.

Proof. Since the restriction $\beta|_U : U \times U \to \Bbbk$ is non-degenerated, we have $U \cap U^\perp = 0$. Moreover, for any $v \in V$ there exists a unique $u_v \in U$ such that $\beta(u, v) = \beta(u, u_v)$ for all $u \in U$. The latter property forces $u - u_v \in U^\perp$. Thus, each $v \in V$ is decomposed as $v = u_v + (u - u_v)$, where $u_v \in U$ and $u - u_v \in U^\perp$. Hence, $U + U^\perp = V$. □

Definition 2.1
Linear mapping $f : V_1 \to V_2$ between vector spaces with scalar products β_1, β_2 is called *an isometry*, if $\forall\, v, w \in V_1 \quad \beta_1(v, w) = \beta_2\big(f(v), f(w)\big)$.

2.6.1 Isotropic and anisotropic subspaces. A bilinear form β on V is called *anisotropic* if $\beta(v, v) \neq 0$ for all non-zero $v \in V$. For example, any Euclidean scalar product in $\mathbb{R}^n$ is anisotropic by the definition. Anisotropy of a symmetric form implies its non-degeneracy. A subspace U of a vector space V with a scalar product β is called *anisotropic*, if the restricted form $\beta|_U$ is anisotropic.

Exercise 2.11. Let $\dim V = 2$, β be a scalar product on V, and B be the Gram matrix of β in some basis of V. Show that β on V is anisotropic iff $-\det B$ is not a square in $\Bbbk$.

A subspace $U \subset V$ is called *isotropic* for a bilinear form β on V, if $\beta|_U \equiv 0$, i.e.

$$\forall\, u_1, u_2 \in U \quad \beta(u_1, u_2) = 0\,.$$

In particular, non-zero vectors v such that $\beta(v, v) = 0$ are called *isotropic vectors*. If β is a non-degenerated scalar product on V, then it follows from prop. 2.10 that $\dim U \leqslant \dim V/2$ for any isotropic subspace $U \subset V$.

Example 2.6 (Hyperbolic space H_{2n})
Let $\dim V = n$. Equip $H_{2n} \stackrel{\text{def}}{=} V^* \oplus V$ with a scalar product

$$h\big(\,(\xi_1, v_1)\,,\,(\xi_2, v_2)\,\big) = \xi_1(v_2) + \xi_2(v_1)\,. \tag{2-10}$$

[1] possibly degenerated

Thus, h is restricted to the zero form on both V, V^*, and for all $v \in V$ and $\xi \in V^*$

$$h(\xi, v) = h(v, \xi) = \langle \xi , v \rangle .$$

In a basis of H combined form dual bases $e_1^*, e_2^*, \dots, e_n^* \in V^*$, $e_1, e_2, \dots, e_n \in V$ as

$$(e_1^*, 0),\ (e_2^*, 0),\ \dots ,\ (e_n^*, 0),\ (0, e_1),\ (0, e_2),\ \dots ,\ (0, e_n) \in V^* \oplus V , \tag{2-11}$$

the Gram matrix of h looks like

$$H = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} , \tag{2-12}$$

where E and 0 are identity and zero $n \times n$-matrices. Thus, h is non-degenerated and has isotropic subspaces of the maximal possible dimension.

Each basis of H_{2n} with the Gram matrix (2-12) is called *a hyperbolic basis*. To pass from a hyperbolic basis to orthogonal one, it is enough to put $p_i = e_i + e_i^*$ and $q_i = e_i - e_i^*$. Note, that $h(p_i, p_i) = 2$ and $h(q_i, q_i) = -2$.

2.6.2 Orthogonal sums. A direct sum $V_1 \oplus V_2$ of spaces V_1, V_2 with scalar products β_1, β_2 admits a scalar product $\beta_1 \dotplus \beta_2$ that takes (u_1, u_2), $(w_1, w_2) \in V_1 \oplus V_2$ to

$$\left[\beta_1 \dotplus \beta_2\right] \left((u_1, u_2),\ (w_1, w_2)\right) \stackrel{\text{def}}{=} \beta_1(u_1, u_2) + \beta_2(w_1, w_2) . \tag{2-13}$$

We call the sum $V_1 \oplus V_2$ equipped with scalar product (2-13) *an orthogonal* direct sum and denote it by $V_1 \dotplus V_2$.

Exercise 2.12. Show that $H_{2m} \dotplus H_{2k} \simeq H_{2(m+k)}$ (an isometric isomorphism).

Theorem 2.3
Any vector space V with a scalar product splits into orthogonal direct sum of a hyperbolic subspace and an anisotropic subspace.

Proof. Induction in $\dim V$. If there are no isotropic vectors in V (say, if $\dim V = 1$), then V itself is anisotropic. If V contains isotropic vector e, then there is $w \in V$ such that $\beta(e, w) \neq 0$. Vector $u = w / \beta(e, w)$ satisfies $\beta(e, u) = 1$. Put $e^\vee = u - \frac{1}{2}\beta(u, u) \cdot e$. Since $\beta(e, e^\vee) = \beta(e, u) = 1$ and $\beta(e^\vee, e^\vee) = 0$, vectors e and $e^\vee$ span the hyperbolic plane $H_2 \subset V$. Then $V = H_2 \dotplus H_2^\perp$ by prop. 2.12. By induction, $H_2^\perp = H_{2k} \dotplus W$, where W is anisotropic. Thus, $V = H_{2k+2} \dotplus W$. □

Remark 2.1. We will see in theorem 2.6 below that the decomposition from theorem 2.3 is unique up to isometric isomorphisms between orthogonal direct sumands.

Corollary 2.14 (from the proof of theorem 2.3)
In any vector space V with a scalar product β any two vectors $u, v \in V$ such that u is isotropic and $\beta(u, v) \neq 0$ span a hyperbolic plane $H_2 \subset V$.

Corollary 2.15
Each k-dimensional isotropic subspace of any space V with a scalar product is contained in some hyperbolic subspace $H_{2k} \subset V$.

Proof. Induction in k. The case $k = 1$ is covered by cor. 2.14. For $k > 1$ consider a hyperbolic plane $H_2 \subset V$, spanned by any non zero vector $u \in U$ and any $v \in V$ such that $\beta(u,v) \neq 0$. An intersection $U' = U \cap H_2^\perp = U \cap v^\perp$ is a hyperplane in U and is transversal to u. Thus, U' is $(k-1)$-dimensional isotropic subspace in $H_2^\perp$. By induction, U' is contained in some hyperbolic subspace $H_{2k-2} \subset H_2^\perp$. Hence, $U \subset H_2 \dotplus H_{2k-2} = H_{2k} \subset V$. □

Corollary 2.16

The following properties of a vector space V with a scalar product β are equivalent:

1) V is isometrically isomorphic to a hyperbolic space
2) V is a direct sum of two isotropic subspaces
3) $\dim V$ is even and there is an isotropic subspace $U \subset V$ of dimension $\dim U = \dim V/2$.

Proof. Implication (1) ⟹(2) is obvious. Let (2) hold. By cor. 2.15 dimensions of both isotropic components do not exceed $\dim V/2$. This forces both dimensions to be equal $\dim V/2$. Thus, (2) ⟹(3). Implication (3) ⟹(1) follows from cor. 2.15 : hyperbolic subspace of V that contains an isotropic subspace of dimension $\dim V/2$ has to coincide with V. □

2.6.3 Action of isometries. Choose a basis in V and write B for the Gram matrix of the scalar product β. If an isometry $f : V \to V$ has matrix F, then

$$F^t \cdot B \cdot F = B\,. \tag{2-14}$$

Since $\det B \neq 0$, (2-14) implies $\det F \neq 0$. Thus, each isometry is invertible and all the isometries of V form a group. It is called *orthogonal group* (or *isometry group*) of β and is denoted by $\mathrm{O}_\beta(V)$. It also follows from (2-14) that the matrix of f^{-1} is $F^{-1} = B^{-1}F^tB$.

Example 2.7 (isometries of hyperbolic plane)

Pick up a standard hyperbolic basis e, e^* of H_2 with the Gram matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Then an isometric operator should have a matrix $F = \begin{pmatrix}a&b\\c&d\end{pmatrix}$ that satisfies $\begin{pmatrix}a&c\\b&d\end{pmatrix}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, which means $ac = bd = 0$ and $ad + bc = 1$. These equations have two families of solutions:

$$F_\lambda = \begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix} \quad\text{and}\quad \widetilde{F}_\lambda = \begin{pmatrix}0&\lambda\\\lambda^{-1}&0\end{pmatrix}\,, \quad \text{where } \lambda \in \Bbbk \smallsetminus \{0\}. \tag{2-15}$$

When $\Bbbk = \mathbb{R}$, operators F_λ with $\lambda > 0$ are called *a hyperbolic rotations*, because an orbit of generic vector $v = (x,y)$ under the action of F_λ is a hyperbola $xy = \text{const}$ and if we put $\lambda = e^t$ and pass to orthogonal basis

$$p = \sqrt{\tfrac{1}{2}}\,(e + e^*)\ , \quad q = \sqrt{\tfrac{1}{2}}\,(e - e^*)\ ,$$

then F_λ will have a matrix

$$\begin{pmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{pmatrix}\cdot\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix}\cdot\begin{pmatrix}1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&-1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}\cosh t&\sinh t\\\sinh t&\cosh t\end{pmatrix}$$

similar to the matrix of euclidean rotation. For $\lambda < 0$ isometry F_λ is a hyperbolic rotation composed with central symmetry about the origin[1]. All operators F_λ are proper, i.e. preserve orientation. Operators $\widetilde{F}_\lambda$ are non-proper and can be described as compositions of hyperbolic rotations with reflection w.r.t the axis of hyperbolas.

Example 2.8 (reflections in hyperplanes)
Each anisotropic vector $e \in V$ provides an orthogonal decomposition $V = \Bbbk \cdot e \dotplus e^\perp$. A mapping $\sigma_e : V \to V$ that inverts e and identically acts on $e^\perp$ (see fig. 2⋄11) is called *a reflection* in hyperplane $e^\perp$. Clearly, $\sigma_e \in \mathrm{O}_\beta(V)$ and $\sigma_e^2 = \mathrm{Id}_V$. The action of σ_e on an arbitrary $v \in V$ is

$$\sigma_e(v) = v - 2\,\frac{\beta(e,v)}{\beta(e,e)} \cdot e\,. \tag{2-16}$$

Exercise 2.13. Make it sure and prove that $f \circ \sigma_e \circ f^{-1} = \sigma_{f(e)}$ for any isometry $f : V \to V$ and any anisotropic $e \in V$.

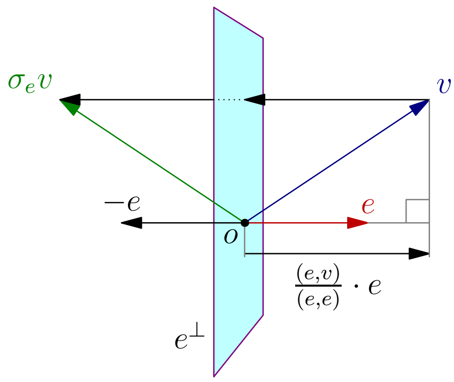

Рис. 2⋄11. Reflection σ_e.

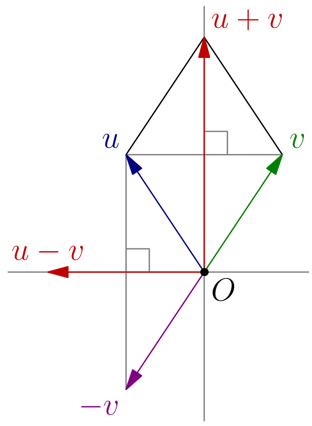

Рис. 2⋄12. Reflections in rombus.

Lemma 2.2
Let V be an arbitrary space with a scalar product β and $u, v \in V$ have $\beta(u,u) = \beta(v,v) \neq 0$. Then there is a reflection in a hyperplane that sends u either to v or to $-v$.

Proof. If u and v are collinear, then $\sigma_v = \sigma_u$ is a required reflection. If u and v span a plane, then at least one of the diagonals $u+v$, $u-v$ of the rhombus on fig. 2⋄12 is anisotropic, because they are perpendicular: $\beta(u+v, u-v) = \beta(u,u) - \beta(v,v) = 0$ and span the same plane as u, v, which is not isotropic by assumptions. Reflection σ_{u-v} takes $u \mapsto v$, reflection σ_{u+v} takes $u \mapsto -v$. □

Theorem 2.4
Each isometry of an arbitrary n-dimensional space with a scalar product is a composition of at most $2n$ reflections in hyperplanes.

[1]with projective viewpoint, as λ goes through 0 or ∞ the moving point $F_\lambda v$ crosses infinite line and appears on the other branch of the same hyperbolic orbit

Proof. Induction in n. The orthogonal group of 1-dimensional space is exhausted by Id and reflection $-\mathrm{Id}$. Let $n > 1$ and $f : V \to V$ be an isometry. Pick up some anisotropic $v \in V$ and write σ for a reflection that sends $f(v)$ either to v or to $-v$. Then σf tekes v either to v or to $-v$ and maps the hyperplane $v^\perp$ to itself. By induction, the restriction of σf onto $v^\perp$ is a composition of at most $2n-2$ reflections in some hyperplanes of $v^\perp$. Extend them to the reflections of V by including v into the mirror hyperplane of each reflection. The composition of resulting reflections of V coincides with σf on the hyperplane $v^\perp$ and acts on v either as σf (if $\sigma f(v) = v$) or as $-\sigma f$ (if $\sigma f(v) = -v$). Thus, $\sigma f : V \to V$ is the composition of $2n-2$ reflections just described and, maybe, one more reflection in the hyperplane $v^\perp$. Hence $f = \sigma\sigma f$ is the composition of at most $2n$ reflections. □

Theorem 2.5 (Witt lemma)
Let U_1, W_1, U_2, W_2 be spaces with scalar products. If some two spaces in triple U_1, $U_1 \dotplus W_1$, W_1 are isometrically isomorphic to the corresponding two spaces in triple U_2, $U_2 \dotplus W_2$, W_2, then the remaining third elements of triples are isometrically isomorphic as well.

Proof. If there are isomorphic isomorphisms $f : U_1 \xrightarrow{\sim} U_2$ and $g : W_1 \xrightarrow{\sim} W_2$, then their direct sum $f \oplus g : U_1 \dotplus W_1 \to U_2 \dotplus W_2$ sends $(u, w) \in U_1 \dotplus W_1$ to $\big(f(u), g(w)\big) \in U_2 \dotplus W_2$ and gives the required isometry. Two remaining cases are completely symmetric, it is enough to consider one of them. Assume that there are isomorphic isomorphisms

$$f : U_1 \xrightarrow{\sim} U_2 \quad \text{and} \quad h : U_1 \dotplus W_1 \xrightarrow{\sim} U_2 \dotplus W_2 .$$

Using induction in $\dim U_1 = \dim U_2$, let us show that there exists an isometry $g : W_1 \xrightarrow{\sim} W_2$. If $U_1 = \Bbbk \cdot u$ has dimension 1, vector u is anisotropic. Hence $f(u) \in U_2$ and $h(u, 0) \in U_2 \dotplus W_2$ are anisotropic as well and have the same scalar squares as u. Write σ for a reflection of $U_2 \dotplus W_2$ taking $h(u, 0) \mapsto \big(\pm f(u), 0\big)$. Composition $\sigma h : U_1 \dotplus W_1 \xrightarrow{\sim} U_2 \dotplus W_2$ isometrically sends U_1 onto U_2. Hence, it isometrically identifies their orthogonal complements, that is, gives a required isometry $\sigma h|_{W_1} : W_1 \xrightarrow{\sim} W_2$.

If $\dim U_1 > 1$, pick up some anisotropic $u \in U_1$ and consider orthogonal direct sums

$$U_1 \dotplus W_1 = \Bbbk \cdot u \dotplus u^\perp \dotplus W_1 \quad \text{and} \quad U_2 \dotplus W_2 = \Bbbk \cdot f(u) \dotplus f(u)^\perp \dotplus W_2 ,$$

where $u^\perp \subset U_1$ and $f(u)^\perp \subset U_2$ are the orthogonal complements to anisotropic vectors u and $f(u)$ inside U_1 and U_2 respectively. We have proven already that the existence of isometries

$$h : \Bbbk \cdot u \dotplus u^\perp \dotplus W_1 \xrightarrow{\sim} \Bbbk \cdot f(u) \dotplus f(u)^\perp \dotplus W_2 , \quad f : U_1 \xrightarrow{\sim} U_2 , \quad f|_{\Bbbk \cdot u} : \Bbbk \cdot u \xrightarrow{\sim} \Bbbk \cdot f(u)$$

implies the existence of some isometries $f' : u^\perp \xrightarrow{\sim} f(u)^\perp$ and $h' : u^\perp \dotplus W_1 \xrightarrow{\sim} f(u)^\perp \dotplus W_2$. Then, by induction, there exists an isometry $W_1 \xrightarrow{\sim} W_2$. □

Theorem 2.6
For any two orthogonal decompositions $V = H_{2k} \dotplus U = H_{2m} \dotplus W$ from theorem 2.3, where U, W are anisotropic and H_{2k}, H_{2m} are hyperbolic, the equality $k = m$ holds and there exists an isometric isomorphism $U \simeq W$.

Proof. Let $m \geqslant k$. Then $H_{2m} = H_{2k} \dotplus H_{2(m-k)}$. The identity mapping $\mathrm{Id} : V \to V$ gives isometry $H_{2k} \dotplus U \xrightarrow{\sim} H_{2k} \dotplus H_{2(m-k)} \dotplus W$. By the Witt lemma there exists an isometry $U \xrightarrow{\sim} H_{2(m-k)} \dotplus W$. Since U is anisotropic, the hyperbolic space $H_{2(m-k)}$ should vanish, because otherwise it contains isotropic vector. Thus, $k = m$ and there is an isometry $U \xrightarrow{\sim} W$. □

Corollary 2.17
Isometry group of V acts transitively on the set of all hyperbolic subspaces of any fixed dimension.

Proof. Let $H_1, H_2 \subset V$ be hyperbolic subspaces of the same dimension. Since we have isometries $f : H_1 \simeq H_2$ and $\mathrm{Id}_V : H_1 \dotplus H_1^\perp \simeq H_2 \dotplus H_2^\perp$, there should be an isometry $g : H_1^\perp \simeq H_1^\perp$. Then $f \oplus g : H_1 \dotplus H_1^\perp \to H_2 \dotplus H_2^\perp$ is an isometry of V and sends H_1 to H_2 as required. □

Corollary 2.18
Isometry group of V acts transitively on the set of all isotropic subspaces of any fixed dimension.

Proof. By cor. 2.15 and cor. 2.17 it is enough to check that for any two maximal isotropic subspaces U, W in the hyperbolic space $V = H_{2n}$ there exist an isometry of V sending U to W.

If $U \cap W = 0$, then $V = U \oplus W$ and the correlation operator of β induces an isomorphism $\widehat{\beta}|_U : U \hookrightarrow W^*$, which takes vector $u \in U$ to covetor $w \mapsto \beta(w, u)$. Thus, there are dual bases $u_1, u_2, \dots, u_n \in U$, $w_1, w_2, \dots, w_n \in W$ such that

$$\beta(w_i, u_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j\,. \end{cases}$$

Then $u_1, u_2, \dots, u_n$, $w_1, w_2, \dots, w_n$ is a hyperbolic basis for V and operator $f : V \to V$ that sends $u_i \mapsto w_i$, $w_j \mapsto u_j$ is isometric and exchanges U with W.

If $\dim U \cap W > 0$, we proceed by induction in $\dim U \cap W$ like in cor. 2.15. Consider a hyperbolic plane $H_2 \subset V$ spanned by some $v_1 \in U \cap W$ and $v_2 \in V$ such that $\beta(v_1, v_2) \neq 0$. The intersections $U' = U \cap H_2^\perp = U \cap v_2^\perp$ and $W' = W \cap H_2^\perp = W \cap v_2^\perp$ have codimension 1 in U and W respectively and both are transversal to v_1. Thus U' and W' are $(n-1)$-dimensional isotropic subspaces in $H_2^\perp$. Their intersection $U' \cap W'$ is a hyperplane in $U \cap W$ and is transversal to v_1 as well. By induction, there is an isometry of $H_2^\perp$ that sends U' to W'. The direct sum of this isometry with the identity map $\mathrm{Id} : H_2 \to H_2$ gives the required isometry of V taking U to W. □

Corollary 2.19
The group of all linear projective isomorphisms $\mathbb{P}_n \simeq \mathbb{P}_n$ preserving a smooth quadric $Q \subset \mathbb{P}_n$ acts transitively on the set of all projective subspaces of any fixed dimension laying on Q (in particular, on the points of the quadric).

Example 2.9 (real quadratic forms)
Any quadratic form over $\mathbb{R}$ can be written in appropriate coordinates as[1]

$$q(x) = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+m}^2\,. \tag{2-17}$$

Ordered pair (p, m) is called *a signature* of q. Each pair of basic vectors e_i, e_{p+i} corresponding to the opposite coefficients $(+1, -1)$ in (2-17) spans a hyperbolic plane with hyperbolic basis

$$e = (e_i + e_{p+i})/\sqrt{2}\,, \quad e^* = (e_i + e_{p+i})/\sqrt{2}\,.$$

Thus, $\min(p, m)$ equals the dimension of maximal isotropic subspace of q. We conclude that in each dimension d there are precisely two non-isometric anisotropic real spaces with scalar products: *positively defined*, which has $m = 0$, and *negatively defined*, which has $p = 0$.

[1]take any basis with diagonal Gram matrix and divide each basic vector e_i with $q(e_i) \neq 0$ by $\sqrt{|q(e_i)|}$

The difference $\operatorname{ind}(q) \stackrel{\text{def}}{=} p - m$ is called *an index* of q. It does not depend on choice of basis, because $|\operatorname{ind}(q)|$ equals the dimension of anisotropic component of q and $\operatorname{ind} q > 0$ iff those component is positive. Since the sum $p + m = \operatorname{rk} q$ also does not depend on a choice of basis, we conclude that the whole signature does not as well.

Forms of opposite indexes produce the same quadric $V(q)$. Absolute value $|\operatorname{ind}(q)|$ is called *an index* of real quadric $V(q)$. Since all smooth quadrics of fixed signature are projectively equivalent[1], let us write $Q_{n,m}$ for a smooth quadric of signature $(n+1-m, m)$ in $\mathbb{P}_n = \mathbb{P}(\mathbb{R}^{n+1})$, assuming that $m \leqslant (n+1)/2$.

Proposition 2.13
Real smooth quadric $Q_{n,m} \subset \mathbb{P}_n$ is $(m-1)$-planar.

Proof. Induction in m. If $m = 0$, the quadric is empty and has planarity -1. If $m > 0$, then prop. 2.11 and cor. 2.12 imply that planarity of $Q_{n,m}$ is one more than planarity of quadric $Q_{n-2,m-1}$ that appears as an intersection of Q with projective subspace $x_0 = x_1 = 0$, which lies in the tangent space to $Q_{n,m}$ at the point $e_0 + e_1 \in Q_{n,m}$ and is complementary to this point. $\square$

Corollary 2.20
Quadrics $Q_{n,m}$ form a complete list of pairwise non-equivalent real projective quadrics. $\square$

Example 2.10 (quadratic forms over $\mathbb{F}_p = \mathbb{Z}/(p)$, $p > 2$)
Squaring non-zero elements: $x \mapsto x^2$ is an endomorphism of the multiplicative group $\mathbb{F}_p^*$. Its kernel consists of two elements ± 1, because quadratic polynomial $x^2 = 1$ has exactly two roots $x = \pm 1$ in a field $\mathbb{F}_p$. Thus, proper squares form a subgroup of index 2 in $\mathbb{F}_p^*$ and $\mathbb{F}_p^*/\mathbb{F}_p^{*\,2} \simeq \{\pm 1\}$.

Let us pick up some non-square $\varepsilon \in \mathbb{F}_p^* \smallsetminus \mathbb{F}_p^{*\,2}$ once forever. Then any non-square has a form $\vartheta^2\varepsilon$. Hence, in dimension 1 non-degenerated quadratic forms over $\mathbb{F}_p$ are exhausted up to isometry by anisotropic forms x^2 и εx^2.

In dimension 2 each quadratic form $ax_1^2 + bx_2^2$ maps $\mathbb{F}_p^2$ to $\mathbb{F}_p$ *surjectively*. Indeed, for any given $c \in \mathbb{F}_p$ the sets $\{a\,x_1^2 \mid x_1 \in \mathbb{F}_p\}$ and $\{c - b\,x_2^2 \mid x_2 \in \mathbb{F}_p\}$ are of the same cardinality $(p+1)/2$. Hence they have at least one common element $a\vartheta_1^2 = c - b\vartheta_2^2$, for which $a\vartheta_1^2 + b\vartheta_2^2 = c$.

Exercise 2.14. Deduce from this that over $\mathbb{F}_p$ any space of dimension $\geqslant 3$ with a scalar product contains an isotropic vector.

Thus, anisotropic spaces over $\mathbb{F}_p$ exist only in dimensions 1, 2. Let us list them in dim 2.

Since there exists a vector e with $\beta(e,e) = 1$, each non-degenerate quadratic form of dimension 2 in appropriate coordinates turns to either $x_1^2 + x_2^2$ or $x_1^2 + \varepsilon x_2^2$. These two forms are non-isomorphic, because they have distinct determinants modulo squares.

Form $x_1^2 + x_2^2$ is hyperbolic[2] iff its discriminant $D/4 = -1$ is a square. By Fermat's little theorem, the subgroup of squares in $\mathbb{F}_p^*$ is contained in the kernel of the endomorphism $\gamma : x \mapsto x^{\frac{p-1}{2}}$ of the multiplicative group $\mathbb{F}_p^*$. Endomorphism γ is non-trivial, because polynomial $x^{(p-1)/2} = 1$ has at most $(p-1)/2$ roots, and $\operatorname{im}\gamma = \{\pm 1\}$, because $\gamma(x)^2 = x^{p-1} = 1$ for all $x \in \mathbb{F}_p^*$. Thus, the group of non-zero squares coincides with $\ker\gamma$. In particular, -1 is a square iff $(-1)^{(p-1)/2} = 1$, that is iff $p \equiv 1 \pmod 4$. Thus, form $x_1^2 + x_2^2$ is anisotropic for $p \equiv -1 \pmod 4$ and hyperbolic for $p \equiv 1 \pmod 4$.

[1] i.e. go to each other under appropriate linear projective automorphism of enveloping space
[2] i.e. has isotropic vector

By the same reasons, form $x_1^2 + \varepsilon x_2^2$ is anisotropic for $p \equiv 1 \pmod 4$ and hyperbolic for $p \equiv -1 \pmod 4$.

We conclude that any smooth quadratic form over $\mathbb{F}_p$ either is hyperbolic or is a direct orthogonal sum of a hyperbolic form and one of the following anisotropic forms:

$$x^2\,, \quad \varepsilon x^2\,, \quad x_1^2 + x_2^2 \quad (\text{for } p = 4k+3)\,, \quad \varepsilon x_1^2 + x_2^2 \quad (\text{for } p = 4k+1)\,.$$

Home task problems to §2

In all the problems below the ground field $\Bbbk$ is assumed, by default, to be algebraically closed of char $\Bbbk \neq 2$ except for those problems where $\Bbbk$ is precisely given.

Problem 2.1. Prove that for any two involutions of $\mathbb{P}_1$ there exists a unique pair of points interchanged by the both involutions.

Problem 2.2. Consider an arbitrary pencil of conics containing at least one smooth conic. Could it contain precisely a) 0 b) 1 c) 2 d) 3 e) 4 singular conics? If so, give the examples.

Problem 2.3. Are there two smooth conics intersecting precisely in a) 1 b) 2 c) 3 points? If so, give the examples.

Problem 2.4. How many shared tangent lines may have two smooth conics?

Problem 2.5 (Lamé's theorem). Show that polar lines of a given fixed point $a \in \mathbb{P}_2$ w.r.t. a conic running through some pencil of conics are intersecting at one point.

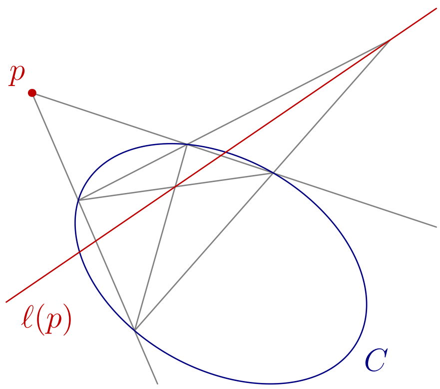

Fig. 2⋄13. Drawing a polar.

Problem 2.6 (Steiner's drawing). Shown on fig. 2⋄13 is Jacob Steiner's method of drawing a polar line $\ell(p)$ of a given point p w.r.t. a given smooth conic C using only a ruler. Explain why does it work.

Problem 2.7. Using only the ruler (and Steiner's method), draw a line passing through a given point p and touching a given conic C. Consider two cases: a) $p \notin C$ b) $p \in C$.

Problem 2.8. Using only the ruler, draw a triangle inscribed in a given smooth conic Q and such that his sides a, b, c pass through 3 given points A, B, C. How many solutions may have this problem?

Problem 2.9. Formulate and solve projectively dual problem to the previous one.

Problem 2.10. Given 4 mutually skew[1] lines in 3D-space, how many lines does intersect them all? Consider the cases when 3D-space in question is: a) $\mathbb{CP}_3$ b) $\mathbb{RP}_3$ c) affine $\mathbb{C}^3$ d) affine $\mathbb{R}^3$.

[1]in projective space this means «non-intersecting», in affine space this means «not laying in a shared plane»

Find all possible answers and indicate those which are stable w.r.t. small perturbation of the 4 given lines.

Problem 2.11. How many solutions have equations a) $x_0^2+x_1^2+x_2^2=0$ b) $x_0^2+x_1^2+x_2^2+x_3^2=0$ c) $x_1^2+x_2^2+x_3^2=-1$ over the field $\mathbb{F}_9$, which consist of 9 elements $a+b\sqrt{-1}$, $a,b=-1,0,1$, added and multiplied modulo 3.

Problem 2.12 (Schläflische Doppelsechs). The double six line configuration of Schäfli is constructed as follows. Let $[0], [1], \dots, [5] \subset \mathbb{P}_3$ be six lines such that $[1], \dots, [5]$ are mutually skew, $[0]$ intersects all of them, and each of $[1], \dots, [5]$ does not either touch or lie on the quadric drown through any 3 other. Show that:

a) $\forall\, i=1,\dots,5\ \exists$ unique line $[i'] \neq [0]$ such that $\forall\, j\neq i\ [i']\cap[j]\neq\varnothing$

b) $\forall\, i=1,\dots,5\ \forall\, j\neq i\ [i']\cap[i]=[i']\cap[j']=\varnothing$

c) each of $[1'],\dots,[5']$ does not either touch or lie on the quadric drown through any 3 other

d) $\exists$ unique line $[0']$ that intersects each of $[1'],\dots,[5']$.

(Hint: let $[0_1']\neq[1]$ and $[0_2']\neq[2]$ intersect all $[1'],\dots,[5']$ except for $[1']$ and $[2']$ respectively; prove that they have the same intersection points p_3, p_4, p_5 with $[3']$, $[4']$, $[5']$ by constructing these points only by means of lines $[3]$, $[4]$, $[5]$, $[3']$, $[4']$, $[5']$, and $[0]$.)

e*) Show that each double six line configuration lies on some smooth cubic surface S, which contains 15 more lines besides the double six.

§3 Tensor Guide

3.1 Tensor products and Segre varieties. Let $V_1, V_2, \dots, V_n$ and W be vector spaces of dimensions $d_1, d_2, \dots, d_n$ and m over an arbitrary field $\Bbbk$. A map

$$\varphi : V_1 \times V_2 \times \cdots \times V_n \to W$$

is called *multilinear*, if it is linear in each argument when all the other remain to be fixed:

$$\varphi(\dots , \lambda v' + \mu v'' , \dots) = \lambda \, \varphi(\dots , v' , \dots) + \mu \, \varphi(\dots , v'' , \dots) .$$

Multilinear maps $V_1 \times V_2 \times \cdots \times V_n \to W$ form a vector space. We denote it by

$$\mathrm{Hom}(V_1, V_2, \dots, V_n; W) .$$

If we fix a basis $\{e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)}\}$ in each V_i and a basis $\{e_1, e_2, \dots, e_m\}$ in W, then any multilinear map $\varphi \in \mathrm{Hom}(V_1, V_2, \dots, V_n; W)$ is uniquely determined by its values on all combinations of the basic vectors:

$$\varphi\big(e_{\alpha_1}^{(1)} , e_{\alpha_2}^{(2)} , \dots , e_{\alpha_n}^{(n)} \big) = \sum_{\nu} a_\nu^{(\alpha_1,\alpha_2,\dots,\alpha_n)} \cdot e^\nu \in W ,$$

i.e. by $m \cdot \prod d_\nu$ constants $a_\nu^{(\alpha_1,\alpha_2,\dots,\alpha_n)} \in \Bbbk$ that can be thought of as entries of some matrix of «$(n+1)$-dimensional format» $m \times d_1 \times d_2 \times \cdots \times d_n$, if you can imagine such a thing[1]. Multilinear map φ presented by such a matrix sends an arbitrary collection of vectors $(v_1, v_2, \dots, v_n)$, in which $v_i = \sum\limits_{\alpha_i=1}^{d_i} x_{\alpha_i}^{(i)} e_{\alpha_i}^{(i)} \in V_i$ for $1 \leqslant i \leqslant n$, to

$$\varphi(v_1, v_2, \dots, v_n) = \sum_{\nu=1}^{m} \Bigg(\sum_{\alpha_1,\alpha_2,\dots,\alpha_n} a_\nu^{(\alpha_1,\alpha_2,\dots,\alpha_n)} \cdot x_{\alpha_1}^{(1)} \cdot x_{\alpha_2}^{(2)} \cdot \cdots \cdot x_{\alpha_n}^{(n)} \Bigg) \cdot e^\nu \in W .$$

Thus, $\dim \mathrm{Hom}(V_1, V_2, \dots, V_n; W) = \dim W \cdot \prod\limits_\nu \dim V_\nu$.

Exercise 3.1. Check that

a) a collection of vectors $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$ does not contain zero vectors iff there exists a multilinear map φ (to somewhere) such that $\varphi(v_1, v_2, \dots, v_n) \neq 0$

b) a composition of any multilinear map $\varphi : V_1 \times V_2 \times \cdots \times V_n \to U$ with any *linear operator* $F : U \to W$ is a multilinear map $F \circ \varphi : V_1 \times V_2 \times \cdots \times V_n \to W$.

3.1.1 Tensor product of vector spaces.

Composing linear maps $F : U \to W$ with fixed multilinear map

$$\tau : V_1 \times V_2 \times \cdots \times V_n \to U , \tag{3-1}$$

we get a linear map

$$\mathrm{Hom}(U, W) \xrightarrow{F \mapsto F \circ \tau} \mathrm{Hom}(V_1, V_2, \dots, V_n; W) . \tag{3-2}$$

Definition 3.1

For given $V_1, V_2, \dots, V_n$ a multilinear map (3-1) is called *universal* if the linear map (3-2) is an isomorphism for *all* vector spaces W.

[1] usual matrices, which present *linear* maps $V \to W$, have just 2-dimensional format $d \times m$

In other words, the multilinear map τ is universal, if for any W and any multilinear map

$$\varphi : V_1 \times V_2 \times \cdots \times V_n \to W$$

there exist a unique linear operator $F : U \to W$ such that $\varphi = F \circ \tau$. Graphically, this means that two *multilinear* solid arrows in diagram

$$\begin{tikzcd}
& U \arrow[dd, dashed, "F"] \\
V_1 \times V_2 \times \cdots \times V_n \arrow[ur, "\tau"] \arrow[dr, "\varphi"'] & \\
& W
\end{tikzcd}$$

can be *always* completed by a *unique* dotted *linear* arrow.

Lemma 3.1
For any two universal multilinear maps $\tau_1 : V_1 \times V_2 \times \cdots \times V_n \to U_1$ and $\tau_2 : V_1 \times V_2 \times \cdots \times V_n \to U_2$ there exists a unique linear isomorphism $\iota : U_1 \to U_2$ such that $\tau_2 = \iota \tau_1$.

Proof. Since both U_1, U_2 are universal, there exist a unique pair of linear operators $F_{21} : U_1 \to U_2$ and $F_{12} : U_2 \to U_1$ fitted in commutative diagrams

$$\begin{tikzcd}
U_1 \arrow[rr, equal, "\mathrm{Id}_{U_1}"] & & U_1 \\
& V_1 \times V_2 \times \cdots \times V_n \arrow[ul, "\tau_1"] \arrow[ur, "\tau_1"'] \arrow[dl, "\tau_2"] \arrow[dr, "\tau_2"'] & \\
U_2 \arrow[uu, dashed, "F_{12}"] \arrow[rr, equal, "\mathrm{Id}_{U_2}"'] & & U_2 \arrow[uu, dashed, "F_{21}"']
\end{tikzcd}$$

$$\begin{tikzcd}
U_1 \arrow[dr, dashed, "F_{21}"] \arrow[dd, equal, "\mathrm{Id}_{U_1}"'] & & & & U_2 \arrow[dl, dashed, "F_{12}"'] \arrow[dd, equal, "\mathrm{Id}_{U_2}"] \\
& U_2 \arrow[dl, dashed, "F_{12}"] & V_1 \times V_2 \times \cdots \times V_n \arrow[ull, "\tau_1"'] \arrow[l, "\tau_2"'] \arrow[dll, "\tau_1"] \arrow[urr, "\tau_2"] \arrow[r, "\tau_1"] \arrow[drr, "\tau_2"'] & U_1 \arrow[dr, dashed, "F_{21}"'] & \\
U_1 & & & & U_2
\end{tikzcd}$$

We claim that $F_{21}F_{12} = \mathrm{Id}_{U_2}$ and $F_{12}F_{21} = \mathrm{Id}_{U_1}$, because the uniqueness of decompositions $\tau_1 = \varphi \circ \tau_1$, $\tau_2 = \psi \circ \tau_2$ forces $\varphi = \mathrm{Id}_{U_1}$, $\psi = \mathrm{Id}_{U_2}$. □

Lemma 3.2
Let us fix for each $1 \leqslant i \leqslant n$ some basis $e_1^{(i)}, e_2^{(i)}, \dots, e_{d_i}^{(i)} \in V_i$ and write $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ for a vector space of dimension $\prod d_i$ whose basis consists of all possible formal tensor products

$$e_{\alpha_1}^{(1)} \otimes e_{\alpha_2}^{(2)} \otimes \ldots \otimes e_{\alpha_n}^{(n)}, \quad 1 \leqslant \alpha_i \leqslant d_i . \tag{3-3}$$

Then a multilinear map $\tau : V_1 \times V_2 \times \cdots \times V_n \to V_1 \otimes V_2 \otimes \cdots \otimes V_n$ that sends each collection

$$(e^{(1)}_{\alpha_1}, e^{(2)}_{\alpha_2}, \dots , e^{(n)}_{\alpha_n}) \in V_1 \times V_2 \times \cdots \times V_n$$

to their formal tensor product (3-3) is universal.

Proof. Given a multilinear map $\varphi : V_1 \times V_2 \times \cdots \times V_n \to W$ and linear $F : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to W$, the identity $\varphi = F \circ \tau$ forces $F(e^{(1)}_{\alpha_1} \otimes e^{(2)}_{\alpha_2} \otimes \dots \otimes e^{(n)}_{\alpha_n}) = \varphi(e^{(1)}_{\alpha_1}, e^{(2)}_{\alpha_2}, \dots , e^{(n)}_{\alpha_n})$ for any collection of basic vectors. $\square$

Definition 3.2
The vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called *tensor product* of $V_1, V_2, \dots , V_n$. The universal multilinear map $\tau : V_1 \times V_2 \times \cdots \times V_n \to V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called *a tensor multiplication* of vectors. We write $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ for the image $\tau(v_1, v_2, \dots , v_n)$ of an arbitrary collection of vectors. Such the tensor products are called *decomposable tensors.*

3.1.2 Segre varieties. Since the tensor multiplication τ is just multilinear but *not linear*, its image, which consists of decomposable tensors, is not a vector subspace in $V_1 \otimes V_2 \otimes \cdots \otimes V_n$. It forms some non-linear subvariety of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ that span $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ as a vector space. The projectivisation of the variety of decomposable tensors is called *the Segre variety*. The tensor multiplication τ induces *the Segre embedding*:

$$s : \mathbb{P}(V_1) \times \mathbb{P}(V_2) \times \cdots \times \mathbb{P}(V_n) \to \mathbb{P}\left(V_1 \otimes V_2 \otimes \cdots \otimes V_n\right) \tag{3-4}$$

that sends a collection of 1-dimensional subspaces $\Bbbk \cdot v_i \subset V_i$ spanned by non zero $v_i \in V_i$ to 1-dimensional subspace $\Bbbk \cdot v_1 \otimes v_2 \otimes \cdots \otimes v_n \subset V_1 \otimes V_2 \otimes \cdots \otimes V_n$.

Exercise 3.2. Check that Segre's mapping (3-4) is a well defined[1] inclusion.

Since the linear span of decomposable tensors is the whole space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, the Segre variety is not contained in a hyperplane. However its dimension, which equals $\sum m_i$, where $m_i = d_i - 1$, is much smaller then $\dim \mathbb{P}\left(V_1 \otimes V_2 \otimes \cdots \otimes V_n\right) = \prod(1+m_i) - 1$. Thus, a «generic» tensor[2] is not decomposable, certainly. By the construction, the Segre variety is ruled by n families of projective subspaces of dimensions $m_1, m_2, \dots , m_n$. The simplest example of the Segre variety is provided by the Segre quadric from n° 2.4.1 on p. 39.

Example 3.1 (isomorphism $U^* \otimes V \simeq \mathrm{Hom}(U, V)$ and decomposable linear maps)
For any two vector spaces U, W there is a bilinear map $U^* \times W \to \mathrm{Hom}(U, V)$ sending a pair $(\xi, w) \in U^* \times W$ to linear mapping $U \to W$ that takes $u \mapsto \langle\, \xi , u \,\rangle \cdot w$. By the universality of the tensor multiplication, there exists a unique linear mapping

$$U^* \otimes V \to \mathrm{Hom}(U, V) \tag{3-5}$$

taking $\xi \otimes w$ to the linear operator $u \mapsto \langle\, \xi , u \,\rangle \cdot w$. This operator has rank 1, its image is spanned by $w \in W$, its kernel is a hyperplane $\mathrm{Ann}\,(\xi) \subset U$.

Exercise 3.3. Check that each linear mapping $F : U \to W$ of rank 1 equals $\xi \otimes w$ for appropriate choice of $\xi \in U^*$, $w \in W$uniquely predicted by F up to proportionality.

[1] i.e. tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ is non zero and it is changed by proportional one when each v_i is changed by some proportional vector

[2] in any reasonable sense

Exercise 3.4. For U and V of finite dimensions prove that linear mapping (3-5) is an isomorphism.

Geometrically, rank 1 operators form the Segre variety $S \subset \mathbb{P}_{mn-1} = \mathbb{P}(\mathrm{Hom}(U, W))$. Its linear span is the whole of $\mathbb{P}(\mathrm{Hom}(U, W))$ and it is ruled by two families of projective spaces $\xi \otimes \mathbb{P}(W)$ and $\mathbb{P}(U^*) \otimes w$.

Using matrix entries (a_{ij}) as homogeneous coordinates in $\mathbb{P}(\mathrm{Hom}(V, W))$, we can describe the Segre variety by a system of quadratic equations

$$\det \begin{pmatrix} a_{ij} & a_{ik} \\ a_{\ell j} & a_{\ell k} \end{pmatrix} = a_{ij}a_{\ell k} - a_{ik}a_{\ell j} = 0$$

saying that $\operatorname{rk} A = 1$. The Segre embedding

$$\mathbb{P}(U^*) \times \mathbb{P}(V) = \mathbb{P}_{n-1} \times \mathbb{P}_{m-1} \hookrightarrow \mathbb{P}_{mn-1} = \mathbb{P}(\mathrm{Hom}(U, W)),$$

takes a pair of points $x = (x_1 : x_2 : \cdots : x_n)$, $y = (y_1 : y_2 : \cdots : y_n)$ to rank 1 matrix ${}^t y \cdot x$ whose $a_{ij} = x_j y_i$. For $\dim U = \dim W = 2$ we get exactly the Segre quadric in $\mathbb{P}_3$ considered in n° 2.4.1 on p. 39.

3.2 Tensor algebra and contractions.

Tensor product

$$V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n}$$

is called n th *tensor power* of V. We also put

$$V^{\otimes 0} \stackrel{\text{def}}{=} \Bbbk \quad \text{and} \quad V^{\otimes 1} \stackrel{\text{def}}{=} V.$$

All the tensor powers are combined in the infinite dimensional associative non commutative graded algebra

$$TV \stackrel{\text{def}}{=} \bigoplus_{n \geqslant 0} V^{\otimes n}$$

whose multiplication is the tensor multiplication. Algebra TV is called *the tensor algebra* of V. A choice of basis $e_1, e_2, \dots, e_n \in V$ identifies TV with an algebra of polynomials in n *non-commuting* variables e_ν. Namely, lemma 3.2 implies that the tensor monomials

$$e_{\nu_1} \otimes e_{\nu_2} \otimes \cdots \otimes e_{\nu_m} \tag{3-6}$$

form a basis of TV over $\Bbbk$ and are multiplied by writing after one other consequently and putting $\otimes$ between. This multiplication is extended onto linear combinations of monomials by the usual distributivity rules.

Algebra TV is also called *the free associative* $\Bbbk$*-algebra* spanned by V. This name outlines the following universal property of linear inclusion

$$\iota : V \hookrightarrow TV \tag{3-7}$$

putting V into TV as the component $V^{\otimes 1}$. For any associative $\Bbbk$-algebra A and any $\Bbbk$-linear mapping of vector spaces $f : V \to A$ there exists a unique homomorphism of associative algebras

$\alpha : TV \to A$ such that $f = \alpha \circ \iota$. In other words, algebra homomorphisms $TV \to A$ stay in bijection with linear mappings $V \to A$ for all algebras A.

Exercise 3.5. Follow the proof of lemma 3.1 to show that the free associative algebra and the inclusion (3-7) are defined by the universal property uniquely up to unique isomorphism of associative algebras commuting with the inclusions (3-7). Also check that the tensor algebra (3-7) does satisfy the universal property.

3.2.1 Dualities and total contraction. There is a canonical pairing between

$$\left(V^*\right)^{\otimes n} \quad \text{and} \quad V^{\otimes n}$$

provided by *complete contraction* that takes $\xi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$ and $v = v_1 \otimes v_2 \otimes \cdots \otimes v_n$ to

$$\langle\, \xi \,,\, v \,\rangle \stackrel{\text{def}}{=} \prod_{i=1}^{n} \left\langle\, \xi_i \,,\, v_i \,\right\rangle . \tag{3-8}$$

Since R.H.S. is multilinear in v_i's, each collection of ξ_i's produces well defined linear mapping $V^{\otimes n} \to \Bbbk$ whose dependence on ξ_i's is multilinear. This gives linear map $V^{*\otimes n} \to \left(V^{\otimes n}\right)^*$. If V is of finite dimension, then there are dual bases $e_1, e_2, \dots, e_n \in V$, $x_1, x_2, \dots, x_n \in V^*$ and the basic tensor monomials $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}$ and $x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_s}$ clearly form the dual bases for pairing (3-8). Thus, for finite dimensional V we have canonical isomorphism

$$\left(V^{\otimes n}\right)^* \simeq \left(V^*\right)^{\otimes n} . \tag{3-9}$$

The universal property of $V^{\otimes n}$ produces another tautological duality: space $\left(V^{\otimes n}\right)^*$ of all linear maps $V^{\otimes n} \to \Bbbk$ is canonically isomorphic to the space of n-linear forms $\underbrace{V \times V \times \cdots \times V}_{n} \to \Bbbk$:

$$\left(V^{\otimes n}\right)^* \simeq \operatorname{Hom}(V, \dots, V\, ;\, \Bbbk) . \tag{3-10}$$

Combining (3-9) and (3-10) together, we get canonical isomorphism

$$\left(V^*\right)^{\otimes n} \simeq \operatorname{Hom}(V, \dots, V\, ;\, \Bbbk) , \tag{3-11}$$

that sends decomposable tensor $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$ to n-linear form $V \times V \times \cdots \times V \to \Bbbk$ taking

$$(v_1, v_2, \dots, v_n) \mapsto \prod_{i=1}^{n} \xi_i(v_i) .$$

3.2.2 Partial contractions. Let $\{1, 2, \dots, p\} \xleftarrow{I} \{1, 2, \dots, m\} \xrightarrow{J} \{1, 2, \dots, q\}$ be two arbitrary[1] inclusions. As usually, we write i_ν, j_ν instead of $I(\nu)$, $J(\nu)$. We treat the images $I = (i_1, i_2, \dots, i_m)$, $J = (j_1, j_2, \dots, j_m)$ of both inclusions as numbered collections of indexes staying in bijection with each other. Linear mapping

$$c^I_J : V^{*\otimes p} \otimes V^{\otimes q} \to V^{*\otimes(p-m)} \otimes V^{\otimes(q-m)}$$

[1] not necessary monotonous

that contracts i_ν th factor of $V^{*\otimes p}$ with j_ν th factor of $V^{\otimes q}$ for $\nu = 1, 2, \dots, m$ and keeps all the other factors in the same order

$$\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_p \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_q \mapsto \prod_{\nu=1}^{m} \xi_{i_\nu}(v_{j_\nu}) \cdot (\bigotimes_{i \notin I} \xi_i) \otimes (\bigotimes_{j \notin J} v_j) \tag{3-12}$$

is called *a partial contraction* in indexes I, J. Note that different choices of I, J and even different numberings of indexes inside fixed subsets I, J lead to *different* contraction maps.

Example 3.2 (свертка вектора с полилинейной формой)
Let us treat n-linear form $\varphi(v_1, v_2, \dots, v_n)$ as a tensor from $V^{*\otimes n}$ via isomorphism (3-11) and contract it with a vector $v \in V$ in the first tensor factor. We get a tensor from $V^{*\otimes(n-1)}$, which produces an $(n-1)$-linear form on $V \times V \times \cdots \times V$ backwards via (3-11). This form is called *internal product* of v and φ and is denoted by $i_v\varphi$ or by $v \llcorner \varphi$.

Exercise 3.6. Check that internal multiplication by v is nothing else as fixation of v as the first argument in φ, that is, $i_v\varphi(w_1, w_2, \dots, w_{n-1}) = \varphi(v, w_1, w_2, \dots, w_{n-1})$.

3.2.3 Linear support of a tensor. For any subspaces $U, W \subset V$ we have a coincidence

$$(U \cap W)^{\otimes n} = U^{\otimes n} \cap W^{\otimes n} \quad \text{inside } V^{\otimes n}$$

which is evident in the standard monomial basis of $V^{\otimes n}$ constructed from a basis

$$e_1, \dots, e_p, u_1, \dots, u_q, w_1, \dots, w_r, v_1, \dots, v_s \in V$$

where e_i form a basis in $U \cap W$, u_j and w_k complete it to some bases in U, W, and v_m complete all this stuff to a basis for V.

Thus, for any tensor $t \in V^{\otimes n}$ there is a *minimal* subspace $\operatorname{Supp}(t) \subset V$ whose n th tensor power contains t. It is called *a linear support* of t and coincides with the intersection of all $W \subset V$ such that $t \in W^{\otimes n}$. We put $\operatorname{rk} t \overset{\text{def}}{=} \dim \operatorname{Supp} t$ an call it *a rank* of t. We say that tensor t is *degenerated* if $\operatorname{rk} t < \dim V$. In this case there exist a linear change of coordinates that reduces the number of variables in t. For example, if $\dim \operatorname{Supp}(t) = 1$, then $t = c \cdot v^{\otimes n}$ for some $c \in \Bbbk$ and $v \in V$.

We are going to describe $\operatorname{Supp}(t)$ more constructively as a linear span of some finite set of vectors effectively computed from t. For each injective (not necessary monotonous) map

$$J : \{1, 2, \dots, (n-1)\} \hookrightarrow \{1, 2, \dots, n\} \tag{3-13}$$

write $J = \{j_1, j_2, \dots, j_{n-1}\} \subset \{1, 2, \dots, n\}$ for its image, put $\hat{\jmath} = \{1, 2, \dots, n\} \smallsetminus J$, and consider contraction map

$$c_t^J : V^{*\otimes(n-1)} \to V, \quad \xi \mapsto c^{(1,2,\dots,(n-1))}_{(j_1, j_2, \dots, j_{n-1})}(\xi \otimes t) \tag{3-14}$$

that couples ν th tensor factor of $V^{*\otimes(n-1)}$ with j_ν th tensor factor of t for all $1 \leqslant \nu \leqslant (n-1)$. The result of this contraction is a linear combination of $\hat{\jmath}$ th tensor factors of t. Clearly, it lies in $\operatorname{Supp}(t)$.

Theorem 3.1
Linear support $\operatorname{Supp}(t)$ of any $t \in V^{\otimes n}$ is spanned by the images of contractions (3-14) for all possible choices of numbered indexes (3-13) being contracted.

Proof. Let $\mathrm{Supp}(t) = W$. We have to check that each linear form $\xi \in V^*$ annihilating all im $\left(c_t^J\right)$ has to annihilate W as well. Assume the contrary: let a linear form $\xi \in V^*$ annihilate all $c_t^J\left(V^{*\otimes(n-1)}\right)$ but have non-zero restriction on W. Chose a basis $\xi_1, \xi_2, \dots, \xi_d \in V^*$ such that $\xi_1 = \xi$ and the restrictions of $\xi_1, \xi_2, \dots, \xi_k$ onto W form a basis in W^*. Expand t through the tensor monomials in the dual basis $w_1, w_2, \dots, w_k \in W$. Then $\xi\left(c_t^J(\xi_{\nu_1} \otimes \xi_{\nu_2} \otimes \cdots \otimes \xi_{\nu_{n-1}})\right)$ equals the complete contraction of t with the basic monomial $\xi_1 \otimes \xi_{\nu_1} \otimes \xi_{\nu_2} \otimes \cdots \otimes \xi_{\nu_{n-1}}$, where the coupling of tensor factors is prescribed by J. The result of this contraction coincides with the coefficient at the dual basic monomial in our expansion of t. In this way we can reach a coefficient at any monomial of t that contains w_1. Thus, all these coefficient vanish and w_1 does not appear in t, i.e. $w_1 \notin \mathrm{Supp}(t)$. Contradiction. □

3.3 Symmetric and Grassmannian algebras.

A multilinear map

$$\varphi : \underbrace{V \times V \times \cdots \times V}_{n} \to U \tag{3-15}$$

is called *symmetric* if it does not change its value under any permutation of the arguments. We call φ *skew symmetric* if it vanishes each time when some two of its arguments coincide.

Exercise 3.7. Show that under a permutation of arguments the value of skew symmetric multilinear map is multiplied by the sign of the permutation and this property is equivalent to skew symmetry when char $\Bbbk \neq 2$, .

Symmetric and skew-symmetric multilinear maps (3-15) form vector subspaces in $\mathrm{Hom}(V, \dots, V; U)$. We denote them as

$$\mathrm{Sym}^n(V, U) \subset \mathrm{Hom}(V, \dots, V; U) \quad \text{and} \quad \mathrm{Skew}^n(V, U) \subset \mathrm{Hom}(V, \dots, V; U)\,.$$

Linear mapping (??), which composes linear operators $F : U \to W$ with a given multilinear map $\varphi : \underbrace{V \times V \times \cdots \times V}_{n} \to U$, respects (skew) symmetry of φ and provides us with a linear mapping from $\mathrm{Hom}(U, W)$ to either $\mathrm{Sym}^n(V, W)$ or $\mathrm{Skew}^n(V, W)$) as soon φ is either symmetric or skew-symmetric. We call φ *universal* (*skew*) *symmetric* multilinear map, if the corresponding linear mapping $F \mapsto F \circ \varphi$ is an isomorphism *for all* W.

The universal symmetric multilinear map is denoted by

$$\sigma : \underbrace{V \times V \times \cdots \times V}_{n} \to S^n V \tag{3-16}$$

and called *commutative multiplication* of vectors. Its target space $S^n V$ is called n th *symmetric power* of V. Commutative product $\sigma(v_1, v_2, \dots, v_n)$ is usually denoted by $v_1 \cdot v_2 \cdot \cdots \cdot v_n$ or $v_1 v_2 \dots v_n$.

The universal skew-symmetric multilinear map is denoted by

$$\alpha : \underbrace{V \times V \times \cdots \times V}_{n} \to \Lambda^n V \tag{3-17}$$

and called *exterior*[1] product of vectors. Its target space $\Lambda^n V$ is called n th *exterior power* of V. Skew product $\alpha(v_1, v_2, \dots, v_n)$ is usually denoted by $v_1 \wedge v_2 \wedge \cdots \wedge v_n$.

Exercise 3.8. Show that universal multilinear maps (3-16) and (3-17), if exist, are unique up to unique isomorphism of their target spaces commuting with the universal maps.

The existence of the universal (skew) symmetric multilinear maps follows from the existence of the tensor product. Their target spaces are factors of tensor power $V^{\otimes n}$ by (skew) symmetry relations. To get them for all n at once, it is better to factorize the whole of free associative algebra $T(V)$ by appropriate double sided ideal, which takes it to a free (skew) commutative algebra.

3.3.1 Symmetric algebra. Write $\mathcal{I}_{\text{sym}} \subset TV$ for a double sided ideal generated by a vector subspace of $V \otimes V$ spanned by all the differences

$$u \otimes w - w \otimes u\,, \quad u, w \in V\,. \tag{3-18}$$

By the definition, a homogeneous component of degree n in $\mathcal{I}_{\text{sym}}$, that is an intersection $\mathcal{I}_{\text{sym}} \cap V^{\otimes n}$, is a linear span of all differences

$$(\cdots \otimes v \otimes w \otimes \cdots) - (\cdots \otimes w \otimes v \otimes \cdots)\,, \tag{3-19}$$

where the both terms are decomposable of degree n and differ only in order of v, w. Then the whole ideal

$$\mathcal{I}_{\text{sym}} = \bigoplus_{n \geqslant 0} \left(\mathcal{I}_{\text{sym}} \cap V^{\otimes n}\right)\,.$$

Factor algebra $SV \stackrel{\text{def}}{=} TV/\mathcal{I}_{\text{sym}}$ is called *a symmetric algebra* of V. Its multiplication is induced by the tensor multiplication in TV. It is called *commutative multiplication* and denoted by dot or by nothing. The symmetric algebra is graded

$$SV = \bigoplus_{n \geqslant 0} S^n V\,, \quad \text{where} \quad S^n V \stackrel{\text{def}}{=} V^{\otimes n}/(\mathcal{I}_{\text{sym}} \cap V^{\otimes n})\,.$$

A choice of basis $e_1, e_2, \dots, e_d \subset V$ identifies SV with polynomial algebra $\Bbbk[e_1, e_2, \dots, e_d]$ in variables e_i. Under this identification the symmetric powers turn to subspaces of homogeneous polynomials $S^n V \subset \Bbbk[e_1, e_2, \dots, e_d]$ in agreement with n° 1.3.1 on p. 9.

Exercise 3.9. Find $\dim S^n V$.

Proposition 3.1
Commutative multiplication (i.e. tensor multiplication followed by factorization through $\mathcal{I}_{\text{sym}}$)

$$\sigma : \underbrace{V \times V \times \cdots \times V}_{n} \overset{\tau}{\dashrightarrow} V^{\otimes n} \overset{\pi}{\twoheadrightarrow} S^n(V) \tag{3-20}$$

is the universal multilinear map (3-16).

Proof. Any multilinear map $\varphi : V \times V \times \cdots \times V \to W$ is uniquely decomposed as $\varphi = F \circ \tau$, where $F : V^{\otimes n} \to W$ is linear. F is factorizable through π iff $F(\cdots \otimes v \otimes w \otimes \cdots) = F(\cdots \otimes w \otimes v \otimes \cdots)$. But this equivalent to $\varphi(\dots, v, w, \dots) = \varphi(\dots, w, v, \dots)$. □

[1] as well as *grassmannian*, or *skew-commutative*, or *super-commutative*

Exercise 3.10. Verify that SV is *a free commutative algebra* spanned by V. Namely, for any linear mapping $f : V \to A$ to commutative $\Bbbk$-algebra A prove that there exists a unique homomorphism of algebras $\tilde{f} : SV \to A$ such that $f = \widetilde{\varphi} \circ \iota$, where $\iota : V \hookrightarrow SV$ embeds V into SV as the first degree polynomials. Show that $\iota : V \hookrightarrow SV$ is defined by this universal property uniquely up to unique isomorphism commuting with ι.

3.3.2 Exterior algebra[1] of V is defined as $\Lambda V \stackrel{\text{def}}{=} TV/\mathcal{I}_{\text{skew}}$, where $\mathcal{I}_{\text{skew}} \subset TV$ is a double sided ideal generated by all tensor squares

$$v \otimes v \in V \otimes V\,. \tag{3-21}$$

Exercise 3.11. Check that $\mathcal{I}_{\text{skew}} \cap V^{\otimes 2}$ contains all sums $v \otimes w + w \otimes v$, $v, w \in V$ and that $\mathcal{I}_{\text{skew}} \cap V^{\otimes 2}$ is spanned by these sums, if $\operatorname{char} \Bbbk \neq 2$.

Like in the symmetric case, the ideal $\mathcal{I}_{\text{skew}}$ is a direct sum of its homogeneous components

$$\mathcal{I}_{\text{skew}} = \bigoplus_{n\geqslant 0} \left(\mathcal{I}_{\text{skew}} \cap V^{\otimes n}\right)\,,$$

where the n th degree component $\mathcal{I}_{\text{skew}} \cap V^{\otimes n}$ is spanned by decomposable tensors of the form $(\cdots \otimes v \otimes v \otimes \cdots)$, $v \in V$. By exrs. 3.11 it contains all sums

$$(\cdots \otimes v \otimes w \otimes \cdots) + (\cdots \otimes w \otimes v \otimes \cdots) \tag{3-22}$$

as well and is spanned by these sums, if $\operatorname{char} \Bbbk \neq 2$. Thus, the exterior algebra is graded:

$$\Lambda V = \bigoplus_{n\geqslant 0} \Lambda^n V\,, \quad \text{where} \quad \Lambda^n V = V^{\otimes n}/(\mathcal{I}_{\text{skew}} \cap V^{\otimes n})\,.$$

Exercise 3.12. Verify that tensor multiplication followed by factorization through $\mathcal{I}_{\text{skew}}$

$$\alpha : \underbrace{V \times V \times \cdots \times V}_{n} \xrightarrow{\tau} V^{\otimes n} \xrightarrow{\pi} \Lambda^n(V) \tag{3-23}$$

is a universal multilinear map (3-17).

Multiplication in ΛV is called *exterior*[2] and is denoted by $v_1 \wedge v_2 \wedge \cdots \wedge v_n$. By exrs. 3.11, under a permutation of factors Grassmannian product of vectors is multiplied by the sign of the permutation:

$$\forall\, g \in \mathfrak{S}_k \quad v_1 \wedge v_2 \wedge \cdots \wedge v_k = \operatorname{sgn}(g) \cdot v_{g_1} \wedge v_{g_2} \wedge \cdots \wedge v_{g_k}\,.$$

3.3.3 Grassmannian polynomials. A choice of basis $e_1, e_2, \dots, e_d \in V$ identifies ΛV with *grassmannian polynomial algebra* $k\,\langle e_1, e_2, \dots, e_d\rangle$ in variables e_i which *skew commute*:

$$\forall\, i, j \quad e_i \wedge e_j = -e_j \wedge e_i\,.$$

These relations force grassmannian monomials to be at most linear in each variable. Thus, any homogeneous grassmannian monomial of degree n is equal up to a sign to some

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}\,, \tag{3-24}$$

where $I = (i_1, i_2, \dots, i_n)$ and $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d$.

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[1] also called *grassmannian algebra* or *a free super-commutative algebra*

[2] as well as *grassmannian* or *skew* or *super-commutative*

Lemma 3.3
Grassmannian monomials (3-24) form a basis of $\Lambda^n V$ as I runs through all collections of n strictly increasing indexes $(i_1, i_2, \dots, i_n) \subset (1, 2, \dots, d)$. In particular, $\Lambda^n V = 0$ for $n > \dim V$, $\dim \Lambda^n V = \binom{d}{n}$, and $\dim \Bbbk \langle e_1, e_2, \dots, e_d \rangle = 2^d$, where $d = \dim V$.

Proof. Write U for a vector space of dimension $\binom{d}{n}$ whose basis consist of formal symbols ξ_I, where I runs through all collections of n strictly increasing indexes $(i_1, i_2, \dots, i_n) \subset (1, 2, \dots, d)$. Define a multilinear map $\alpha : V \times V \times \cdots \times V \to U$ by sending $(e_{j_1}, e_{j_2}, \dots, e_{j_n}) \mapsto \operatorname{sgn}(\sigma) \cdot \xi_I$, where $I = (j_{\sigma(1)}, j_{\sigma(2)}, \dots, j_{\sigma(n)})$ is *strictly increasing* permutation of indexes $(j_1, j_2, \dots, j_n)$. If some of j_ν's coincide, we put $\alpha(e_{j_1}, e_{j_2}, \dots, e_{j_n}) \overset{\text{def}}{=} 0$. Then for any skew-symmetric multilinear map $\varphi : V \times V \times \cdots \times V \to W$ there exists a unique linear operator $F : U \to W$ such that $\varphi = F \circ \alpha$: its action on the basis of U is predicted by the rule $F(\alpha(e_{j_1}, e_{j_2}, \dots, e_{j_n})) = \varphi(e_{j_1}, e_{j_2}, \dots, e_{j_n})$. Thus, α is the universal skew-symmetric multilinear map and there exists an isomorphism $U \xrightarrow{\sim} \Lambda^n V$ that takes $\xi_I \mapsto e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} = e_I$. □

Exercise 3.13. Check that $f(e) \wedge g(e) = (-1)^{\deg(f) \cdot \deg(g)} g(e) \wedge f(e)$ for any homogeneous $f(e), g(e) \in \Bbbk \langle e_1, e_2, \dots, e_d \rangle$ and describe the centre[1] $Z\left(\Bbbk \langle e_1, e_2, \dots, e_d \rangle\right)$.

In the examples below we write $\#I$ for the length n of a collection $I = (i_1, i_2, \dots, i_n)$ of stictly increasing indexes. We also write $|I|$ for the sum $\sum\limits_\nu i_\nu$ and call it *a weight* of I.

$$\hat{I} \overset{\text{def}}{=} \{1, 2, \dots, n\} \smallsetminus I$$

always means the *complementary* collection of strictly increasing indexes.

Exercise 3.14. Check that

$$e_I \wedge e_{\hat{I}} = (-1)^{|I| + \frac{1}{2}\#I(1+\#I)} \cdot e_1 \wedge e_2 \wedge \cdots \wedge e_d \,. \tag{3-25}$$

Example 3.3 (linear substitution of variables)
Under a linear substitution of variables $e_i = \sum\limits_j a_{ij}\, \xi_j$ the basic grassmannian monomials e_I are expressed through ξ_I as

$$\begin{aligned} e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} &= \Big(\sum_{j_1} a_{i_1 j_1}\, \xi_{j_1}\Big) \wedge \Big(\sum_{j_2} a_{i_2 j_2}\, \xi_{j_2}\Big) \wedge \cdots \wedge \Big(\sum_{j_n} a_{i_n j_n}\, \xi_{j_n}\Big) = \\ &= \sum_{1 \leqslant j_1 < j_2 < \cdots < j_n \leqslant n} \; \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, a_{i_1 j_{\sigma(1)}} a_{i_2 j_{\sigma(2)}} \cdots a_{i_n j_{\sigma(n)}}\, \xi_{j_1} \wedge \xi_{j_2} \wedge \cdots \wedge \xi_{j_n} = \sum_J a_{IJ}\, \xi_J \,, \end{aligned}$$

where J runs through all collections of $\#J = n$ increasing indexes and a_{IJ} stays for $n \times n$-minor of matrix (a_{ij}) situated in rows $i_1, i_2, \dots, i_n$ and columns $j_1, j_2, \dots, j_n$,.

Example 3.4 (Laplace expansions)
Let us substitute $e_i = \sum\limits_j a_{ij}\, \xi_j$ in the both sides of identity (3-25). By example 3.3, L.H.S. of (3-25) turns to $\Big(\sum\limits_{\substack{K: \\ \#K = \#I}} a_{IK}\, \xi_K\Big) \wedge \Big(\sum\limits_{\substack{L: \\ \#L = (d - \#I)}} a_{\hat{I}L}\, \xi_L\Big) = (-1)^{\frac{1}{2}\#I(1+\#I)} \sum\limits_{\substack{K: \\ \#K = \#I}} (-1)^{|K|} a_{IK} a_{\hat{I}\hat{K}}\, \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_d$

[1] i. e. all grassmannian polynomials that commute with all grassmannian polynomials

and R.H.S. of (3-25) turns to $(-1)^{\frac{1}{2}\#I(1+\#I)}(-1)^{|I|}\det(a_{ij})\cdot\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_d$. Thus, for any fixed collection $I=(i_1,i_2,\dots,i_n)$ of rows in any square matrix $A=(a_{ij})$ the following relation holds

$$\sum_{\substack{K:\\ \#K=\#I}}(-1)^{|K|+|I|}a_{IK}\hat{a}_{IK}=\det(a_{ij})\,, \tag{3-26}$$

where we write $\hat{a}_{IK}\overset{\text{def}}{=}a_{\hat{I}\hat{K}}$ for the $(d-n)\times(d-n)$-minor complementary[1] to a_{IK} and summation runs through all $(n\times n)$-minors a_{IK} situated in the fixed collection of rows $(i_1,i_2,\dots,i_n)$.

If we repeat the computation replacing $\hat{I}$ by another collection $\hat{J}$, which is complementary to some $J\neq I$, then we get $e_I\wedge e_{\hat{J}}=0$ in R.H.S. of (3-25). This leads to the relation

$$\sum_{\substack{K:\\ \#K=\#I}}(-1)^{|K|+|I|}a_{IK}\hat{a}_{JK}=0\,. \tag{3-27}$$

Identities (3-26) and (3-27) are known as *Laplace relations*. If we number the collections of n increasing indexes in some way and form square matrix $A^{(n)}\overset{\text{def}}{=}\left(a_{IJ}\right)$ of size $\binom{d}{n}\times\binom{d}{n}$, then all the Laplace relations can be written as one matrix identity

$$A^{(n)}\cdot\widehat{A}^{(n)}=\det(a_{ij})\cdot E\,,$$

where $\widehat{A}^{(n)}$ has (IJ)-entry $(-1)^{|I|+|J|}\hat{a}_{JI}$.

Example 3.5 (reduction of grassmannian quadratic forms)
Of course, a grassmannian quadratic form can not be «diagonalized». However, over *any* field $\mathbb{k}$ each homogeneous grassmannian polynomial $\omega\in\Lambda^2V$ can be written in appropriate coordinates in much more convenient form

$$\omega=\xi_1\wedge\xi_2+\xi_3\wedge\xi_4+\cdots+\xi_{2r-1}\wedge\xi_{2r}\,, \tag{3-28}$$

similar to hyperbolic form in symmetric case. Form (3-28) is called *canonical* form of ω and coordinates ξ_i are called *canonical* or *symplectic* coordinates for ω.

To pass from an arbitrary basis e_i to canonical one let us firstly renumber basic vectors in order to have

$$\omega(e)=e_1\wedge(\alpha_2e_2+\cdots+\alpha_ne_n)+e_2\wedge(\beta_3e_3+\cdots+\beta_ne_n)+(\text{terms without } e_1,e_2)\,,$$

where $\alpha_2\neq0$. Then pass to a new basis $\{e_1,\xi_2,e_3,\dots,e_n\}$ that has

$$\xi_2=\alpha_2e_2+\cdots+\alpha_ne_n\,.$$

Substituting $e_2=\left(\xi_2-\beta_3e_3-\cdots-\beta_ne_n\right)/\alpha_2$ in ω, we get

$$\begin{aligned}\omega=e_1\wedge\xi_2+\xi_2\wedge(\gamma_3e_3+\cdots+\gamma_ne_n)+(\text{члены без } e_1 \text{ и } \xi_2)=\\ =(e_1-\gamma_3e_3-\cdots-\gamma_ne_n)\wedge\xi_2+(\text{terms without } e_1,\xi_2)\,.\end{aligned}$$

Then pass to a basis $\{\xi_1,\xi_2,e_3,\dots,e_n\}$, where $\xi_1=e_1-\gamma_3e_3-\cdots-\gamma_ne_n$. We get $\omega=\xi_1\wedge\xi_2+$ (terms without ξ_1,ξ_2) and continue by induction.

Exercise 3.15. Let $A=\left(a_{ij}\right)$ be a skew symmetric matrix. Show that number r in canonical representation of grassmannian quadratic form and $\omega(e)=\sum\limits_{ij}a_{ij}\,e_i\wedge e_j$ does not depend on a choice of symplectic basis and equals[2].

[1]that is situated in the complementary rows and columns
[2]this implies, in particular, that rk A is always even rk A

In the rest of §3 we always assume that char($\Bbbk$)=0.

3.4 Symmetric and skew-symmetric tensors. The symmetric group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the tensors factor of decomposable tensors:

$$g(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{g(1)} \otimes v_{g(2)} \otimes \cdots \otimes v_{g(n)} \quad \text{for } g \in \mathfrak{S}_n\,. \tag{3-29}$$

Since R.H.S. is multilinear in $v_1, v_2, \dots, v_n$ this map is uniquely extended to a linear mapping

$$g : V^{\otimes n} \to V^{\otimes n}\,.$$

Definition 3.3
A tensor $t \in V^{\otimes n}$ is called *symmetric*, if $g(t) = t$ for all $g \in \mathfrak{S}_n$. A tensor $t \in V^{\otimes n}$ is called *skew-symmetric*, if $g(t) = \mathrm{sgn}(g) \cdot t$ for all $g \in \mathfrak{S}_n$. We write

$$\begin{aligned} \mathrm{Sym}^n V &= \{\, t \in V^{\otimes n} \mid \sigma(t) = t \quad \forall\, g \in \mathfrak{S}_n \,\} \\ \mathrm{Skew}^n V &= \{\, t \in V^{\otimes n} \mid g(t) = \mathrm{sgn}(g) \cdot t \quad \forall\, g \in \mathfrak{S}_n \,\} \end{aligned}$$

for the *sub*spaces of $V^{\otimes n}$ formed by (skew) symmetric tensors.

3.4.1 Standard bases. Pick up some basis $e_1, e_2, \dots, e_d \in V$. As soon symmetric tensor t contains some basic monomial m with non zero coefficient, then t contains with the same coefficients the whole S_n-orbit of m. Hence, a basis of $\mathrm{Sym}^n V$ is formed by *complete symmetric tensors*

$$e_{[m_1, m_2, \dots, m_d]} \overset{\text{def}}{=} \left(\begin{gathered} \text{sum of all tensor products of} \\ m_1 \text{ factors } e_1,\, m_2 \text{ factors } e_2,\, \dots,\, m_d \text{ factors } e_d \end{gathered} \right) \tag{3-30}$$

indexed by all collection of non-negative integers $(m_1, m_2, \dots, m_d)$ such that $\sum\limits_\nu m_\nu = n$.

Exercise 3.16. Make it sure that the sum in R.H.S. of (3-30) consists of $\frac{n!}{m_1!\, m_2! \cdots m_d!}$ terms.

Similarly, a basis of $\mathrm{Skew}^n V$ form *complete skew-symmetric tensors*

$$e_{\langle i_1, i_2, \dots, i_n \rangle} \overset{\text{def}}{=} \sum_{g \in \mathfrak{S}_n} \mathrm{sgn}(g) \cdot e_{i_{g(1)}} \otimes e_{i_{g(2)}} \otimes \cdots \otimes e_{i_{g(n)}} \tag{3-31}$$

(sum of $n!$ items).

3.4.2 Symmetrization and alternation. If char $\Bbbk = 0$, there are simple explicit formulas for $\mathfrak{S}_n$-equivariant projectors of $V^{\otimes n}$ onto $\mathrm{Sym}^n V$ and $\mathrm{Skew}^n V$. These projectors are called *symmetrization* and *alternation* and act as

$$\mathrm{sym}_n(t) = \frac{1}{n!} \sum_{g \in \mathfrak{S}_n} g(t) \;:\quad V^{\otimes n} \to \mathrm{Sym}^n(V) \tag{3-32}$$

$$\mathrm{alt}_n(t) = \frac{1}{n!} \sum_{g \in \mathfrak{S}_n} \mathrm{sgn}(g) \cdot g(t) \;:\quad V^{\otimes n} \to \mathrm{Skew}^n(V) \tag{3-33}$$

Exercise 3.17. For any $t \in V^{\otimes n}$, $s \in \mathrm{Sym}^n(V)$, and $a \in \mathrm{Skew}^n(V)$ verify for each $n \geqslant 2$ the relations: a) $\mathrm{sym}_n(t) \in \mathrm{Sym}^n(V)$ b) $\mathrm{alt}_n(t) \in \mathrm{Skew}^n(V)$ c) $\mathrm{sym}_n(s) = s$ d) $\mathrm{alt}_n(a) = a$ e) $\mathrm{sym}_n(a) = \mathrm{alt}_n(s) = 0$.

Example 3.6
For $n = 2$ symmetrization and alternation provide splitting

$$V^{\otimes 2} = \mathrm{Sym}^2(V) \oplus \mathrm{Skew}^2(V)\,. \tag{3-34}$$

Indeed, since each decomposable tensor can be written as

$$u \otimes w = \frac{u \otimes w + w \otimes u}{2} + \frac{u \otimes w - w \otimes u}{2} = \mathrm{sym}_2(u \otimes w) + \mathrm{alt}_2(u \otimes w)\,,$$

images of sym_2 and alt_2 span $V^{\otimes 2}$. They have zero intersection, because

$$\mathrm{sym}_2 \circ \mathrm{alt}_2 = \mathrm{alt}_2 \circ \mathrm{sym}_2 = 0\,.$$

If we treat elements of $V^{\otimes 2}$ as bilinear forms on V^*, splitting (3-34) provides canonical decomposition of an arbitrary bilinear form into a sum of symmetric and skew-symmetric parts:

$$\varphi(x,y) = \frac{\varphi(x,y) + \varphi(y,x)}{2} + \frac{\varphi(x,y) - \varphi(y,x)}{2}\,.$$

Example 3.7
A dimension counting shows that for $n = 3$ a generic tensor $t \in V^{\otimes 3}$ does not lie in $\mathrm{Sym}^3(V) \oplus \mathrm{Skew}^3(V)$. To describe the direct complement to this sum in $V^{\otimes 3}$, write $T : V^{\otimes 3} \to V^{\otimes 3}$ for operator provided by the cyclic permutation $1 \mapsto 2 \mapsto 3 \mapsto 1$ and $E = T^3$ — for the identity. Then an operator

$$p \overset{\text{def}}{=} E - \mathrm{sym}_3 - \mathrm{alt}_3 = \left(2\,E - T - T^2\right)/3 \tag{3-35}$$

is a projection, because $p^2 = \left(4E + T^2 + T - 4\,T - 4\,T^2 + 2\,E\right)/9 = \left(2E - T - T^2\right)/3 = p\,.$

Exercise 3.18. Check that $p \circ \mathrm{alt}_3 = \mathrm{alt}_3 \circ p = p \circ \mathrm{sym}_3 = \mathrm{sym}_3 \circ p = 0$ and deduce that $V^{\otimes 3} = \mathrm{Sym}^3(V) \oplus \mathrm{Skew}^3(V) \oplus \mathrm{im}\,(p)\,.$

Image of p admits an elegant description in terms of 3-linear forms on V^*.

Exercise 3.19. Verify that $\mathrm{im}\,(p)$ consist of all 3-linear forms $\varphi : V^* \times V^* \times V^* \to \Bbbk$ that satisfy *the Jacobi identity* $\forall\ \xi,\eta,\zeta \in V^*\ \varphi(\xi,\eta,\zeta) + \varphi(\eta,\zeta,\xi) + \varphi(\zeta,\xi,\eta) = 0$ and give an explicit example of such a form on 2-dimensional V^*.

Proposition 3.2
If $\mathrm{char}(\Bbbk) = 0$, then the restriction of projection $V^{\otimes n} \to S^n V$ onto subspace $\mathrm{Sym}^n \subset V^{\otimes n}$ as well as the restriction of projection $V^{\otimes n} \to \Lambda^n V$ onto subspace $\mathrm{Skew}^n \subset V^{\otimes n}$ both are isomorphisms. They take

$$e_{[m_1,m_2,\dots,m_d]} \mapsto \frac{(m_1 + m_2 + \cdots + m_d)!}{m_1!\cdot m_2!\,\cdots\, m_d!} \cdot e_1^{m_1} e_2^{m_2} \dots e_d^{m_d} \in S^n V \tag{3-36}$$

$$e_{\langle i_1,i_2,\dots,i_d\rangle} \mapsto n!\cdot e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d} \in \Lambda^n V \tag{3-37}$$

Proof. Each of $n!/(m_1!\,m_2!\cdots m_d!)$ terms of the sum (3-30) is projected to $e_1^{m_1} e_2^{m_2} \dots e_d^{m_d}$ and each of $n!$ items of the sum (3-31) is projected to $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$. □

3.4.3 Important warning. Though isomorphisms of prop. 3.2 the *sub*spaces

$$\mathrm{Sym}^n V,\ \mathrm{Skew}^n V \subset V^{\otimes n}$$

should be never confused with the *factor* spaces $S^n V$, $\Lambda^n V$ of $V^{\otimes n}$. When char($\Bbbk$) = p is positive, all symmetric tensors whose degree is a power of p as well as all skew symmetric tensors whose degree is greater than p go to zero under projections $V^{\otimes n} \twoheadrightarrow S^n V$ and $V^{\otimes n} \twoheadrightarrow \Lambda^n V$. Even if char $\Bbbk = 0$ the isomorphisms of prop. 3.2 identify the standard bases of sub- and factor-spaces only up to combinatorial factor that should be taken into account each time you want to transfer some structure from the subspace to the factor space or backwards.

3.5 Polarisation of commutative polynomials. It follows from prop. 3.2 that for each polynomial $f \in S^n V^*$ there exists a unique symmetric tensor $\tilde f \in \mathrm{Sym}^n V^*$ whose class in the factor space $S^n V^*$ equals f. Considered as n-linear form $\tilde f : V \times V \times \ldots \times V \to \Bbbk$, this tensor is called *a complete polarisation* of polynomial f. It allows to treat f as a polynomial function $f : V \to \Bbbk$ that takes

$$f(v) = \tilde f(v, v, \ldots, v) \qquad \forall\, v \in V\,. \tag{3-38}$$

This is exactly the same function that was attached to a polynomial in n° 1.3.1 on p. 9.

Indeed, if we choose a basis $x_1, x_2, \ldots, x_d \in V^*$ and identify SV^* with polynomial algebra $\Bbbk[x_1, x_2, \ldots, x_d]$, then the complete polarisation of basic monomial $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} \in S^n V^*$ equals

$$\tilde f = \frac{m_1!\ m_2!\ \cdots\ m_d!}{n!} \cdot x_{[m_1, m_2, \ldots, m_d]}\,. \tag{3-39}$$

where $x_{[m_1, m_2, \ldots, m_d]}$ is complete symmetric tensor from formula (3-30), p. 62. For any $v \in V$ the full contraction $\left\langle\, x_{[m_1, m_2, \ldots, m_d]}\,,\, v^{\otimes n} \right\rangle$ is a sum of $n!/(m_1! \cdot m_2! \cdots m_d!)$ mutually equal items

$$\langle\, x_1\,,\, v\,\rangle^{m_1} \langle\, x_2\,,\, v\,\rangle^{m_2} \cdot \langle\, x_d\,,\, v\,\rangle^{m_d}\,. \tag{3-40}$$

Hence, the value of *a function* $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$ on $v \in V$ is equal to the number (3-40). This agrees with formula (1-5), p. 9. Note that we have just proven that evaluation rule (1-5) does not depend on a choice of basis.

Example 3.8 (duality)
Restricting the complete contraction between $V^{\otimes m}$ and $V^{*\otimes m}$ onto subspaces of symmetric tensors and using isomorphism from prop. 3.2, one can define canonical non-degenerated pairing between $S^m V$ and $S^m V^*$. By the definition, the result of coupling of $f \in S^n V$ and $g \in S^n V^*$ equals the complete contraction of $\tilde f \in V^{\otimes m}$ and $\widetilde g \in V^{*\otimes m}$.

Exercise 3.20. Check that the basic monomials of the same name built from dual bases of V and V^* are coupled as

$$\langle\, e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}\,,\, x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}\,\rangle = \frac{m_1! \cdot m_2! \cdot \cdots \cdot m_d!}{n!} \tag{3-41}$$

and all the other couplings vanish.

3.5.1 Polars and derivatives. A partial contraction with a given vector $v \in V$ in the first tensor factor provides us with a linear mapping

$$c_v^1 : V^{*\otimes n} \to V^{*\otimes(n-1)}$$

which fixes $v \in V$ as the first argument of multilinear forms $\varphi \in V^{*\otimes n}$. If we apply this mapping to the complete polarization $\tilde{f}$ of a polynomial $f \in S^n(V^*)$ and project the result to $V^{*\otimes(n-1)}$, we get a linear mapping $S^nV^* \to S^{n-1}V^*$ fitted as the bottom horizontal arrow into diagram

$$\begin{tikzcd} V^{*\otimes n} \supset \mathrm{Sym}^{\,n}V^* \arrow[r, "c_v^1"] \arrow[d] & \mathrm{Sym}^{\,n-1}V^* \subset V^{*\otimes(n-1)} \arrow[d] \\ S^nV^* \arrow[r, "\mathrm{pl}_v"] & S^{n-1}V^*. \end{tikzcd}$$

It sends homogeneous polynomial $f(x) = \tilde{f}(x, x, \dots\, x) \in S^n(V^*)$ to polynomial

$$\mathrm{pl}_v f(x) = \tilde{f}(v, x, \dots\, x) \in S^{n-1}(V^*) \tag{3-42}$$

called *a polar* of v w.r.t. f. The polar depends linearly in $f \in S^n(V^*)$ and in $v \in V$ and its degree is one smaller than degree of f.

For $n = 2$ this construction turns to polar map from n° 2.2.3 on p. 30 and takes quadratic form $q \in S^2V^*$ and vector $v \in V$ to equation of the polar hyperplane of v w.r.t. the quadric $V(q) \subset \mathbb{P}(V)$.

Choosing dual bases $e_1, e_2, \dots, e_d \in V$, $x_1, x_2, \dots, x_d \in V^*$ we see that contraction with $e_i \in V$ in the first tensor factor takes complete symmetric tensor $x_{[m_1,m_2,\dots,m_d]}$ to complete symmetric tensor $x_{[m_1,\dots,m_{i-1},m_i-1,m_{i+1},\dots,m_d]}$, which has $(m_i - 1)$ factors x_i, or to zero, if $m_i = 0$. Thus, by formula (3-36) from prop. 3.2,

$$\mathrm{pl}_{e_i} x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} = \frac{m_i}{n}\, x_1^{m_1} \cdots x_{i-1}^{m_{i-1}} x_i^{m_i-1} x_{i+1}^{m_{i+1}} \cdots x_d^{m_d} = \frac{1}{n}\frac{\partial}{\partial x_i}\, x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}\,.$$

Since $\mathrm{pl}_v f$ is linear in v and f, we conclude that the polar of vector $v = \sum \alpha_i e_i$ w.r.t. polynomial f is *a partial derivative* of f along v divided by $\deg f$:

$$\mathrm{pl}_v f = \frac{1}{\deg(f)}\,\partial_v f = \frac{1}{\deg(f)} \sum \alpha_i \frac{\partial f}{\partial x_i}\,.$$

Note that coordinate-free definition of the polar mapping implies that the partial derivation

$$\partial_v = \sum \alpha_i \frac{\partial}{\partial x_i} : S^nV^* \to S^{n-1}V^*$$

does not depend on a choice of dual bases in V and V^*. It is also evident from the definition that the partial derivations commute: $\partial_u\partial_w = \partial_w\partial_u$. Finally, there is the following straightforward but remarkable identity: $\forall\, u, w \in V$, $\forall\, f \in S^nV^*$, and each m in the range $0 \leqslant m \leqslant n$

$$m!\ \frac{\partial^m f}{\partial u^m}(w) = n!\ \tilde{f}(\underbrace{u, u, \dots, u}_{m}, \underbrace{w, w, \dots, w}_{n}) = (n-m)!\ \frac{\partial^{n-m} f}{\partial w^{n-m}}(u)\,. \tag{3-43}$$

Exercise 3.21. Prove the Leibnitz rule: $\partial_v(f \cdot g) = \partial_v(f) \cdot g + f \cdot \partial_v(g)$.

Since the multilinear form $\tilde f$ is symmetric, the arguments in the middle term of (3-43) may be written in any order. To simplify the notations, let us write $\tilde f(u^m, w^{n-m})$ for the value of $\tilde f$ at the collection of m vectors u and $(n-m)$ vectors w staying in any order.

The standard arguments proving the Newton formula for expansion of binomial $(u+w)^n$ lead to the following explicit identity valid for any polynomial $f \in S^n V^* f$ and any vectors $u, w \in V$:

$$\tilde f\left((u+w)^n\right) = \tilde f(u+w,\, u+w,\, \dots,\, u+w) = \sum_{m=0}^{n} \binom{n}{m} \tilde f\left(u^m, w^{n-m}\right) .$$

Using (3-43), it can be rewritten as *the Taylor expansion*:

$$f(u+w) = \sum_{m=0}^{\deg f} \frac{1}{m!}\, \partial_w^m f(u) . \tag{3-44}$$

Note that R.H.S. is symmetric in u, w because of (3-43).

Exercise 3.22. Make it sure that the value of complete polarisation of polynomial $f \in S^n V^*$ at given collection of vectors $v_1, v_2, \dots, v_n \in V$ equals $\tilde f(v_1, v_2, \dots, v_n) = \dfrac{1}{n!} \partial_{v_1} \partial_{v_2} \dots \partial_{v_n} f$.

3.5.2 Linear support of polynomial $f \in S^n V^*$ is defined as an intersection of all subspaces $W \subset V^*$ such that $f \in S^n W$. We write Supp(f) for the linear support of f. Definitely, it coincides with the linear support of complete polarization $\tilde f \in \mathrm{Sym}^n V^*$. By theorem 3.1, the latter coincides with the image of the contraction map $V^{\otimes(n-1)} \to V^*$ that completely contracts each $t \in V^{\otimes(n-1)}$ with some fixed $(n-1)$ tensor factors of $\tilde f$. Since $\tilde f$ is symmetric, the contraction map does not depend neither on a choice of contracted factors nor on their ordering. Clearly, the image of contraction consists of all linear forms that can be obtained as $(n-1)$-fold partial derivatives of f. Thus, Supp(f) is spanned by

$$\frac{\partial^{m_1}}{\partial x_1^{m_1}} \frac{\partial^{m_2}}{\partial x_2^{m_2}} \cdots \frac{\partial^{m_d}}{\partial x_d^{m_d}} f(x) , \quad \text{where} \quad \sum m_\nu = n-1 . \tag{3-45}$$

Since only the monomial $x_1^{m_1} \dots x_{i-1}^{m_{i-1}} x_i^{m_i+1} x_{i+1}^{m_{i+1}} \dots x_d^{m_d}$ of f brings a contribution to the coefficient at x_i in the linear form (3-45), we conclude that for

$$f = \sum_{\nu_1+\dots+\nu_d=n} \frac{n!}{\nu_1!\,\nu_2!\,\cdots\,\nu_d!}\, a_{\nu_1\nu_2\dots\nu_d}\, x_1^{\nu_1} x_2^{\nu_2} \cdots x_d^{\nu_d} , \tag{3-46}$$

the linear form (3-45) is equal to

$$n! \cdot \sum_{i=1}^{d} a_{m_1\dots m_{i-1}(m_i+1)m_{i+1}\dots m_d}\, x_i . \tag{3-47}$$

There are totally $\binom{n+d-2}{d-1}$ such the linear forms[1] produced by a given $f \in S^n V^*$.

[1]the number of non-negative integer solutions $(m_1, m_2, \dots, m_d)$ of equation $m_1 + m_2 + \dots + m_d = n-1$

Proposition 3.3
Over algebraically closed field $\Bbbk$ of zero characteristic a homogeneous polynomial (3-46) is a perfect n th power of a liner form iff $d \times \binom{n+d-2}{d-1}$-matrix built from the coefficients of linear forms (3-47) has rank 1. In this case a linear form φ, which satisfies $\varphi^n = f$, is proportional to the forms (3-47).

Proof. If $f = \varphi^n$, then $\mathrm{Supp}(f) = \Bbbk \cdot \varphi$ has dimension 1 and all forms (3-47) are proportional to φ. Vice versa, if all forms (3-47) are mutually proportional, then $\mathrm{Supp}(f) = \Bbbk \cdot \psi$ for some non-zero $\psi \in V^*$. Then $S^n(\Bbbk \cdot \psi) = \Bbbk \cdot \psi^n$ and $f = \lambda\psi^n$ for some $\lambda \in \Bbbk$. If $\Bbbk$ is algebraically closed, then $f = \varphi^n$ for $\varphi = \sqrt[n]{\lambda} \cdot \psi$. □

3.5.3 Veronese varieties $V(n,k)$. Geometrically, perfect n th powers φ^n of linear forms $\varphi \in V^*$ considered up to proportionality form *the Veronese variety* $V(n,k) \subset \mathbb{P}(S^nV^*)$, which coincides with the image of degree n *Veronese's embedding*

$$\mathbb{P}_k = \mathbb{P}(V^*) \xrightarrow{\ \varphi \mapsto \varphi^n\ } \mathbb{P}(S^nV^*) = \mathbb{P}_N\,, \quad \text{where} \quad N = \binom{n+k}{k} - 1\,,\ \dim V = k+1\,. \tag{3-48}$$

It follows from prop. 3.3 that $V(n,k) \subset \mathbb{P}_N$ is an algebraic projective variety given by a system of quadratic equations that state vanishing of all 2×2-minors in $d \times \binom{n+d-2}{d-1}$ matrix built of the coefficients of the linear forms (3-47). E.g. for $k = 1$ a homogeneous polynomial in 2 variables

$$f(x_0, x_1) = \sum_{k=0}^{n} a_k \cdot \binom{n}{k} \cdot x_0^{n-k} x_1^k$$

coincides with some proper n th power $(\alpha_0 x_0 + \alpha_1 x_1)^n$ iff

$$\mathrm{rk}\begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_n \end{pmatrix} = 1\,, \quad \text{because of} \quad \frac{\partial^{n-1} f}{\partial x_0^{n-i-1} \partial x_1^i} = a_i x_0 + a_{i+1} x_1\,,$$

what agrees with example 1.9 on p. 17 and is equivalent to a system of quadratic equations

$$\det\begin{pmatrix} a_i & a_j \\ a_{i+1} & a_{j+1} \end{pmatrix} = 0$$

in the coefficients a_i of the polynomial f. If f satisfies these equations, then the coefficients of a linear form, which equals n th root of f, satisfy $(\alpha_0 : \alpha_1) = (a_i : a_{i+1})$.

3.5.4 Polars and tangents. Let $S = V(F) \subset \mathbb{P}(V)$ be degree n hypersurface given by equation $F(x) = 0$, $F \in S^nV^*$. An intersection $S \cap \ell$ with a line $\ell = (pq)$ consists of all pints $\lambda p + \mu q \in \ell$ that satisfy an equation $F(\lambda p + \mu q) = 0$. Its L.H.S. is ether identically zero (what means that $\ell \subset S$) or a homogeneous polynomial of degree n in $(\lambda : \mu)$. If $\Bbbk$ is algebraically closed and $\ell \not\subset S$, then $\ell \cap S$ consists of n points counted with multiplicities, where *the multiplicity* of an intersection point $a_i = \alpha_i' p + \alpha_i'' q \in \ell \cap S$ is defined as an exponent s_i of factor

$$\det\begin{pmatrix} \lambda & \mu \\ \alpha_i' & \alpha_i'' \end{pmatrix} = (\alpha_i''\lambda - \alpha_i'\mu)$$

in the irreducible factorization $f(\lambda,\mu) = \prod(\alpha_i''\mu - \alpha_i'\lambda)^{s_i}$ in $\Bbbk[\lambda,\mu]$. The multiplicity of $a_i \in S \cap \ell$ is also called *local intersection index* of S and ℓ at a_i and denoted by $(S,\ell)_{a_i}$. A line ℓ is called *tangent* to S at $a \in \ell \cap S$ if $(S,\ell)_a \geqslant 2$ or $\ell \subset S$.

The Taylor expansion (3-44) shows that the coefficient at $\lambda^{n-m}\mu^m$ in $F(\lambda p + \mu q)$ equals

$$\binom{n}{m}\tilde{f}(p^{n-m}, q^m) = \frac{1}{m!}\frac{\partial^m F}{\partial q^i}(p) = \frac{1}{(n-m)!}\frac{\partial^{n-m} F}{\partial p^{n-m}}(q)\,. \tag{3-49}$$

If $p \in S$, then the Taylor expansion at p starts with

$$F(p + tq) = t\binom{d}{1}\widetilde{F}(p^{n-1}, q) + t^2\binom{d}{2}\widetilde{F}(p^{n-2}, q^2) + \cdots\,.$$

Thus, line (pq) touches S at $p \in S$ iff $\widetilde{F}(p^{n-1}, q) = 0$.

A point $p \in S$ is called *smooth*, if $F(p^{n-1}, x) \not\equiv 0$ as a linear form in x. In this case all points q such that line (pq) touches S at a given point $p \in S$ fill a hyperplane

$$T_pS = \{x \in \mathbb{P}(V) \mid F(p^{n-1}, x) = 0\} \subset \mathbb{P}(V)\,.$$

This hyperplane is called *a tangent space* of S at p.

If $F(p^{n-1}, x) \equiv 0$, the point $p \in S$ is called *singular* and we say that S is *singular* at p. By (3-49), the coefficients of the linear form

$$F(p^{n-1}, x) = \partial_x F(p)$$

are partial derivatives of F computed at p. Thus, $p \in S$ is a singular point of $S = V(F)$ iff all the partial derivatives of F vanish at p. In this case any line passing through p has at lest 2-fold intersection with S at p and the tangent space T_pS, treated as the union of all tangent lines to S at p, coincides with the whole of $\mathbb{P}(V)$.

For $q \notin S$ as well as for smooth $q \in S$ *an apparent contour* of S from viewpoint q is defined as a closure of a set where the tangent lines drown to S from q do touch S. Since line (qy) touches S at $y \in S$ iff $\widetilde{F}(y^{n-1}, q) = 0$, the apparent contour of S viewed from q is cut out of S by degree $(n-1)$ hypersurface

$$\mathrm{pl}_q S \overset{\text{def}}{=} \{y \in \mathbb{P}(V) \mid \widetilde{F}(q, y^{n-1}) = 0\} \tag{3-50}$$

called $(n-1)$ *th degree polar* of S w.r.t. q.

Note, that $F(q, y^{n-1})$ is a non-zero polynomial in y as soon as $q \notin S$ or $q \in S$ is a smooth point of S. Indeed, if $G(y) = \widetilde{F}(y^{n-1}, q) \equiv 0$ as a polynomial in y, then for $y = q$ we get $F(q) = 0$, i.e. $q \in S$. On the other side, since all the partial derivatives of the zero polynomial are zero as well, we conclude that $\widetilde{F}(q^{n-1}, y) = \widetilde{G}(q^{n-2}, y) = \frac{\partial^{n-2}}{\partial q^{n-2}}G(y) \equiv 0$, i.e. q is a singular point of S.

More generally, r *th degree polar* of a hypersurface $S = V(F) \subset \mathbb{P}(V)$ w.r.t. point $q \in \mathbb{P}(V)$ is defined as

$$\mathrm{pl}_q^{n-r} \overset{\text{def}}{=} \left\{y \in \mathbb{P}(V) \mid \widetilde{F}(q^{n-r}, y^r) = 0\right\}\,.$$

If $q \in S$ is smooth, then the 1 st degree polar of S w.r.t. q is the tangent hyperplane T_qS. Then, inductively, r th degree polar of S w.r.t. q is a hypersurface of degree r that passes through q and has the same polars of degrees $< r$ w.r.t. q as the initial hypersurface S. Thus, a quadratic polar is a quadric $Q \ni q$ such that $T_qQ = T_qS$, a cubic polar is degree 3 hypersurface passing through q and having the same tangent plane and quadratic polar at q as S has, and so on.

3.6 Polarization of grassmannian polynomials. Besides the fact that grassmannian polynomial $\omega \in \Lambda V^*$ can not be interpreted as a function on V, the rest of the previous section is valid for the grassmannian polynomials as well. Namely, it follows from prop. 3.2 that over a field of zero characteristic for any homogeneous grassmannian polynomial $\omega \in \Lambda^n V^*$ there exists a unique skew-symmetric n-linear form $\widetilde{\omega} \in \mathrm{Skew}^{\,n} V^* \subset V^{*\otimes n}$ whose class modulo skew-symmetry relations coincides with ω. Skew-symmetric tensor $\widetilde{\omega} \in V^{*\otimes n}$ is called *complete polarization* of grassmannian polynomial ω. By formula (3-37), p. 63, a complete polarization of the basic grassmannian monomial $\omega = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ equals

$$\widetilde{\omega} = \frac{1}{n!}\, e_{\langle i_1, i_2, \dots, i_n\rangle} = \mathrm{alt}_n\left(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}\right) . \tag{3-51}$$

Like in the symmetric case, we can use the complete contraction between $\mathrm{Skew}^{\,n} V^* \subset V^{*\otimes n}$ and $\mathrm{Skew}^{\,n} V \subset V^{\otimes n}$ to provide $\Lambda^n V^*$ and $\Lambda^n V$ with non-degenerated pairing that couples grassmannian polynomials $\omega \in \Lambda^n V^*$ and $\tau \in \Lambda^n V$ to the total contraction $\langle\, \widetilde{\omega}\,,\, \widetilde{\tau}\,\rangle$.

Exercise 3.23. Verify that the basic grassmannian monomials $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ and $x_J = x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n}$ built from dual bases of V and V^* are coupled as

$$\left\langle\, x_J\,,\, e_I \,\right\rangle = \begin{cases} 1/n! & \text{for } I = J \\ 0 & \text{otherwise} \end{cases} \tag{3-52}$$

(where both collections of indexes I, J are increasing).

3.6.1 Grassmannian partial derivatives. As in the symmetric case, *a polarization* w.r.t. given vector $v \in V$ is a linear mapping

$$\mathrm{pl}_v \,:\, \Lambda^n V^* \to \Lambda^{n-1} V^* ,$$

that takes $\omega \in \Lambda^n V^*$ to the class in $\Lambda^{n-1} V^*$ of contraction of $\widetilde{\omega} \in V^{*\otimes n}$ with $v \in V$ in the *first* tensor factor (now the place of contraction does effect on the sign). Polarization fits into commutative diagram

$$\begin{tikzcd} V^{*\otimes n} \supset \mathrm{Skew}^{\,n} V^* \arrow[r, "c_v^1"] \arrow[d] & \mathrm{Skew}^{\,(n-1)} V^* \subset V^{*\otimes (n-1)} \arrow[d] \\ \Lambda^n V^* \arrow[r, "\mathrm{pl}_v"] & \Lambda^{n-1} V^* . \end{tikzcd}$$

We define *grassmannian partial derivative* of polynomial $\omega \in \Lambda^n V^*$ along a vector $v \in V$ by prescription

$$\partial_v \omega \overset{\text{def}}{=} \deg \omega \cdot \mathrm{pl}_v \omega .$$

Since $\mathrm{pl}_v \omega$ is bilinear in v and ω, partial derivative along $v = \sum \alpha_i e_i$ is a linear combination of partial derivatives along basic vectors:

$$\partial_v = \sum \alpha_i\, \partial_{e_i} .$$

If ω does not depend in x_j, then $\partial_{e_j} \omega = 0$, certainly. Thus, non zero contributions to derivatives of basic monomial $\omega = x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n}$ are coming only from $\partial_{e_{i_1}}, \partial_{e_{i_2}}, \dots, \partial_{e_{i_n}}$. It follows from (3-51) that $\partial_{e_{i_1}} x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} = x_{i_2} \wedge x_{i_3} \wedge \dots \wedge x_{i_n}$ irrespective of ordering of $i_1, i_2, \dots, i_n$ (it could be not necessary increasing). Thus the grassmannian partial derivative along the leftmost factor x_{i_1} acts as usual partial derivative $\partial/\partial x_{i_1}$, that is, just kills this factor. Partial differentiation along the next factors leads to extra signs:

$$\begin{aligned}\partial_{e_{i_k}} x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} &= \partial_{e_{i_k}} (-1)^{k-1} x_{i_k} \wedge x_{i_1} \wedge \dots \wedge x_{i_{k-1}} \wedge x_{i_{k+1}} \dots x_{i_n} = \\ &= (-1)^{k-1} \partial_{e_{i_k}} x_{i_k} \wedge x_{i_1} \wedge \dots \wedge x_{i_{k-1}} \wedge x_{i_{k+1}} \dots x_{i_n} = \\ &= (-1)^{k-1} x_{i_1} \wedge \dots \wedge x_{i_{k-1}} \wedge x_{i_{k+1}} \dots x_{i_n} \,.\end{aligned}$$

Thus, grassmannian differentiation along the k th factor from the left acts as $(-1)^{k-1}\partial/\partial x_{i_k}$. This leads to *Grassmannian Leibnitz rule*:

Exercise 3.24. Make it clear that $\partial_v(\omega \wedge \tau) = \partial_v(\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge \partial_v(\tau)$.

Since $\widetilde{\omega}(u, w, *, \dots, *) = -\widetilde{\omega}(w, u, *, \dots, *)$, grassmannian polarization maps do skew commute: $\mathrm{pl}_u \mathrm{pl}_w \omega = -\mathrm{pl}_w \mathrm{pl}_u \omega$. Hence, grassmannian partial derivatives *skew* commute as well:

$$\partial_u \partial_w = -\partial_w \partial_u \,.$$

In particular, $\partial_v^2 \omega \equiv 0$ for all $v \in V$ and $\omega \in \Lambda V^*$.

3.6.2 Linear support of grassmannian polynomial $\omega \in \Lambda^n V$ is defined as an intersection of all subspaces $W \subset V$ such that $\omega \in \Lambda^n W$. We denote it by $\mathrm{Supp}(\omega)$. Like in the symmetric case, $\mathrm{Supp}\,\omega = \mathrm{Supp}\,\widetilde{\omega}$ coincides with the image of the contraction map $V^{\otimes(n-1)} \to V^*$, which completely contracts each $t \in V^{\otimes(n-1)}$ with some fixed $(n-1)$ tensor factors of $\tilde{f}$. Skew symmetry of $\tilde{f}$ implies that such a contraction does not depend up to a sign neither on a choice of contracted factors nor on their ordering. Thus, $\mathrm{Supp}\,\omega \in V$ is spanned by vectors

$$\partial_J \omega = \partial_{j_1} \partial_{j_2} \dots \partial_{j_{n-1}} \omega \,,$$

where $\partial_j = \partial_{x_j}$ and $J = (j_1, j_2, \dots, j_{n-1})$ runs through all collections of $(n-1)$ distinct indexes[1]. Since the contribution to $\partial_J \omega$ comes only from the basic monomials e_I whose $I \supset J$, a linear support of grassmannian polynomial

$$\omega = \sum_I a_I e_I = \sum_{i_1 i_2 \dots i_n} \alpha_{i_1 i_2 \dots i_n} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$$

(where the coefficients $\alpha_{i_1 i_2 \dots i_n}$ are skew symmetric in $i_1, i_2, \dots, i_n$) is spanned by

$$\partial_J \omega = \pm \sum_{i \notin J} \alpha_{j_1 j_2 \dots j_{n-1} i}\, e_i \,. \tag{3-53}$$

Proposition 3.4
Let $\omega = \sum\limits_{i_1 i_2 \dots i_n} \alpha_{i_1 i_2 \dots i_n} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \in \Lambda V$, where all the coefficients $\alpha_{i_1 i_2 \dots i_n}$ are skew symmetric in indexes $i_1, i_2, \dots, i_n$. Then the following conditions are mutually equivalent:

1) $\omega = u_1 \wedge u_2 \wedge \cdots \wedge u_n$ for some $u_1, u_2, \dots, u_n \in V$

2) $u \wedge \omega = 0 \;\; \forall\, u \in \mathrm{Supp}(\omega)$

[1]it is enough to restrict ourself by strictly increasing collections, certainly

3) for any two collections of distinct indexes $i_1, i_2, \dots, i_{m+1}$ and $j_1, j_2, \dots, j_{m-1}$ *the Plücker relation*[1]

$$\sum_{\nu=1}^{m+1} (-1)^{\nu-1} a_{j_1 \dots j_{m-1} i_\nu} a_{i_1 \dots \hat{i}_\nu \dots i_{m+1}} = 0 \tag{3-54}$$

holds.

Proof. (1) means that ω lies in the top degree component $\Lambda^{\dim \operatorname{Supp}(\omega)} \operatorname{Supp}(\omega)$. This is equivalent to (2) by the following general reason.

Exercise 3.25. Check that $\omega \in \Lambda U$ is homogeneous of degree $\dim U$ iff $u \wedge \omega = 0$ for all $u \in U$.

The Plücker relation (3-54) certifies vanishing of the coefficient at $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_{m+1}}$ in $\partial_{j_1 \dots j_{m-1}} \omega \wedge \omega$ (see formula (3-53), p. 70). Since vectors $u = \partial_{j_1 \dots j_{m-1}} \omega$ span $\operatorname{Supp}(\omega)$, these vanishing conditions are equivalent to (2). □

Exercise 3.26. Write down the Plücker relations for grassmannian quadratic form ω in 4 variables and show that such ω is decomposable[2] iff $\omega \wedge \omega = 0$.

Home task problems to §3

Problem 3.1. Let linear mappings $A \in \operatorname{Hom}(U, V) \simeq U^* \otimes V$, $B \in \operatorname{Hom}(V, W) \simeq V^* \otimes W$ be expanded as $A = \sum \alpha_\nu \otimes a_\nu$, $B = \sum \beta_\mu \otimes b_\mu$, where $\alpha_\nu \in U^*$, $a_\nu \in V$, $\beta_\mu \in V^*$, $b_\mu \in W$. Write similar expansion for the composition $B \circ A \in \operatorname{Hom}(U, W) \simeq U^* \otimes W$.

Problem 3.2. Given dual bases $e_1, e_2, \dots, e_n \in V$ and $x_1, x_2, \dots, x_n \in V^*$, describe how does *the Casimir operator* $\sum x_i \otimes e_i \in V^* \otimes V \simeq \operatorname{End}(V)$ act on V.

Problem 3.3. Write $\tau : \operatorname{End}(V) \to \operatorname{End}(V)^*$ for a correlation that takes $\xi \otimes v \in V^* \otimes V = \operatorname{End}(V)$ to a linear form whose value at $\xi' \otimes v' \in V^* \otimes V = \operatorname{End}(V)$ equals $\langle \xi , v' \rangle \cdot \langle \xi' , v \rangle$. Describe the quadratic form q on the space of linear operators $V \to V$ that produce this correlation. Is q smooth? Write down an explicit formula that computes $\tilde{q}(A, B)$ in terms of capital letters A, B and standard operations with matrices but does not use the matrix elements explicitly.

Problem 3.4. For vector spaces U, V of finite dimensions

a) construct canonical isomorphisms $\operatorname{Hom}\big(U \otimes \operatorname{Hom}(U, W) , W \big) \simeq \operatorname{End}\big(\operatorname{Hom}(U, W)\big) \simeq \operatorname{Hom}\big(U , W \otimes \operatorname{Hom}(U, W)^* \big)$

b) describe an element of $\operatorname{Hom}(U, W)$ corresponding to the map $c : U \otimes \operatorname{Hom}(U, W) \to W$ acting on decomposable tensors by the rule $c(u \otimes \varphi) = \varphi(u)$.

c) Is it true that operator $\tilde{c} : U \to \operatorname{Hom}(U, W)^* \otimes W$ corresponding to c is always injective?

Problem 3.5. For vector spaces U, V, W of finite dimensions construct canonical isomorphism

$$\operatorname{End}(U \otimes V \otimes W) \simeq \operatorname{Hom}\big(\operatorname{Hom}(U, V) \otimes \operatorname{Hom}(V, W) , \operatorname{Hom}(U, W)\big)$$

[1] the «hat» in $a_{i_1 \dots \hat{i}_\nu \dots i_{m+1}}$ means that index i_ν is omitted

[2] i.e. equals a skew product of two linear forms

and describe a linear map $\mathrm{Hom}(U,V)\otimes\mathrm{Hom}(V,W)\to\mathrm{Hom}(U,W)$ that corresponds to the identity endomorphism of $U\otimes V\otimes W$.

Problem 3.6 (spinor decomposition). Let $V=\mathrm{Hom}(U_-,U_+)$, where $\dim U_\pm=2$. Show that canonical direct sum decomposition of $V\otimes V$ into symmetric and skew symmetric parts looks like

$$\underbrace{\left(\left(S^2U_-^*\otimes S^2U_+\right)\oplus\left(\Lambda^2U_-^*\otimes\Lambda^2U_+\right)\right)}_{S^2V}\bigoplus\underbrace{\left(\left(S^2U_-^*\otimes\Lambda^2U_+\right)\oplus\left(\Lambda^2U_-^*\otimes S^2U_+\right)\right)}_{\Lambda^2V}. \tag{3-55}$$

Problem 3.7. Write an explicit example of tensor $t\in V^{\otimes 3}$ that can not be represented as a linear combination of symmetric and skew-symmetric tensors.

Problem 3.8 (Aronhold's principle). For a finite dimensional vector space V over a field of zero characteristic show that perfect n th tensor powers $v^{\otimes n}=v\otimes v\otimes\cdots\otimes v$, where $v\in V$, span the subspace of all symmetric tensors $\mathrm{Sym}^n(V)\subset V^{\otimes n}$ and explicitly represent symmetric tensor $u\otimes w\otimes w+w\otimes u\otimes w+w\otimes w\otimes u$, where $u,w\in V$ are non-proportional, as a linear combination of proper tensor cubes.

Problem 3.9. Over any field $\Bbbk$ (of any characteristic) show that the space of quadratic commutativity relations $\mathscr{I}_{\mathrm{sym}}(V^*)\cap V^*\otimes V^*$ and the space of quadratic skew commutativity relations $\mathscr{I}_{\mathrm{skew}}(V)\cap V\otimes V$ are the annihilators of each other w.r.t. the complete contraction between $V^*\otimes V^*$ and $V\otimes V$.

Problem 3.10. Establish the following canonical isomorphisms (valid over any field $\Bbbk$, of any characteristic): a) $\left(\mathrm{Skew}^nV\right)^*\simeq\Lambda^n\left(V^*\right)$ b) $\left(\mathrm{Sym}^nV\right)^*\simeq S^n\left(V^*\right)$.

Problem 3.11. Prove the following Taylor expansion for the polynomial $\det(A)$ on the space of $n\times n$-matrices:

$$\det(\lambda A+\mu B)=\sum_{p+q=n}\lambda^p\mu^q\cdot\mathrm{tr}\left(\Lambda^pA\cdot\Lambda^qB^t\right),$$

where Λ^pA, Λ^qB are the matrices of operators induced by A, B on the spaces of homogeneous grassmannian polynomials of degrees p, q (matrix elements of Λ^pA, Λ^qB are $p\times p$ and $q\times q$ minors of A, B numbered in such a way that complementary minors have equal numbers).

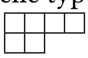
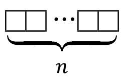
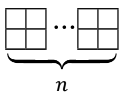
Problem 3.12. Write $S\subset\mathbb{P}_N=\mathbb{P}(S^2V^*)$ for a variety of singular quadrics on $\mathbb{P}_n=\mathbb{P}(V)$. Show that

a) S is an algebraic hypersurface

b) point $Q\in S$ is a smooth point of S iff the corresponding quadric $Q\subset\mathbb{P}_n$ has just one singular point $p\in Q$

c) tangent space $T_QS\subset\mathbb{P}_N$ at a smooth point $Q\in S$ consists of all quadrics in $\mathbb{P}_n$ passing through the singular point of the quadric $Q\subset\mathbb{P}_n$.

Problem 3.13. Let $N:V\to V$ be a nilpotent operator. Describe a cyclic type[1] of operator $N\otimes N:V^{\otimes 2}\to V^{\otimes 2}$ in terms of the cyclic type of N. If generic case seems too difficult, let N be of cyclic type

a)  b)  c) 

Problem 3.14. Compute eigenvalues of $F^{\otimes n}$ for a given diagonal operator $F:V\to V$ with eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$. Use this to analise, are there some non-zero constants $\mu_1,\mu_2,\dots,\mu_m\in\Bbbk$

[1]list of lengths of all Jordan chains in non-increasing order

such that $\forall\, n\in\mathbb{N}\ \ \lambda_1 F_1^{\otimes n}+\lambda_2 F_2^{\otimes n}+\dots+\lambda_m F_m^{\otimes n}=0$ for a given collection of non-zero linear operators $F_1, F_2, \dots , F_m$.

Problem 3.15. For any linear endomorphism $F : V \to V$ make it sure that there are well defined operators $S^k F : S^k V \to S^k V$ and $\Lambda^k F : \Lambda^k V \to \Lambda^k V$ sending decomposable elements to

$$F(v_1\cdot v_2\cdot\ \cdots\ \cdot v_k)=F(v_1)\cdot F(v_2)\cdot\ \cdots\ \cdot F(v_k)$$
$$F(v_1\wedge v_2\wedge\ \cdots\ \wedge v_k)=F(v_1)\wedge F(v_2)\wedge\ \cdots\ \wedge F(v_k)\,.$$

For a diagonal operator F express eigenvalues of $S^n F$ and $\Lambda^n F$ in terms of eigenvalues of F and prove the following identities in the ring $\Bbbk[[t]]$ (assuming that char $\Bbbk = 0$):

a) $\dfrac{1}{\det(E-tF)}=\sum\limits_{k\geqslant 0}\operatorname{tr}\big(S^k F\big)\cdot t^k$ b) $\det(E+tF)=\sum\limits_{k=0}^{\dim V}\operatorname{tr}\big(\Lambda^k F\big)\cdot t^k$.

Problem 3.16. Prove for any square matrix A the identity $e^{A\otimes E+E\otimes A}=e^A\otimes e^A$, where E is the unit matrix.

Problem 3.17. In the ring $\mathbb{Q}[[\dots a_{ij}\dots]]$ of formal power series in n^2 matrix elements a_{ij} of $n\times n$ matrix A prove the identity $\ln\det(E-A)=\operatorname{tr}\ln(E-A)$ and show that for small enough $a_{ij}\in\mathbb{C}$ it holds numerically as well.

Problem 3.18 (De Rahm's and Koszul's complexes). Choose a basis $e_1, e_2, \dots , e_n \in V$ and write $x_i \in SV$, $\xi_i \in \Lambda V$ for the classes of e_i in symmetric and exterior algebras respectively. Let $A=\Lambda V\otimes SV$. Consider two linear mappings: *the De Rahm differential* $d=\sum \xi_\nu\otimes\frac{\partial}{\partial x_\nu} : A\to A$ that takes $\omega\otimes f\mapsto \sum\limits_\nu \xi_\nu\wedge\omega\otimes\frac{\partial f}{\partial x_\nu}$ and *the Koszul differential* $\partial=\sum\frac{\partial}{\partial\xi_\nu}\otimes x_\nu : A\to A$ that takes $\omega\otimes f\mapsto\sum\limits_\nu\frac{\partial\omega}{\partial\xi_\nu}\otimes x_\nu\cdot f$.

a) Show that d and ∂ do not depend on a choice of basis and satisfy $d^2=0$, $\partial^2=0$.

b) Compute $d\partial+\partial d$.

c) (Poincare lemma) Show that both *homology spaces* $\ker d/\operatorname{im} d$ and $\ker\partial/\operatorname{im}\partial$ are 1-dimensional, exhausted by the classes of constants $\Bbbk\cdot 1\otimes 1\subset A$.

§4 Grassmannians

4.1 Plücker quadric and Gr(2,4). Let V be 4-dimensional vector space. The set of all 2-dimensional vector subspaces $U \subset V$ is called *the grassmannian* $\mathrm{Gr}(2,4) = \mathrm{Gr}(2,V)$. More geometrically, the grassmannian $\mathrm{Gr}(2,V)$ is the set of all lines $\ell \subset \mathbb{P}_3 = \mathbb{P}(V)$. Sending 2-dimensional vector subspace $U \subset V$ to 1-dimensional subspace $\Lambda^2 U \subset \Lambda^2 V$ or, equivalently, sending a line $(ab) \subset \mathbb{P}(V)$ to $\Bbbk \cdot a \wedge b \subset \Lambda^2 V$, we get *the Plücker embedding*

$$\mathfrak{u} : \mathrm{Gr}(2,4) \hookrightarrow \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)\,. \tag{4-1}$$

Its image consists of all *decomposable*[1] grassmannian quadratic forms $\omega = a \wedge b$, $a, b \in V$. Clearly, any such a form has zero square: $\omega \wedge \omega = a \wedge b \wedge a \wedge b = 0$. Since an arbitrary form $\xi \in \Lambda^2 V$ can be written[2] in appropriate basis of V either as $e_1 \wedge e_2$ or as $e_1 \wedge e_2 + e_3 \wedge e_4$ and in the latter case ξ is not decomposable, because of $\xi \wedge \xi = 2\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0$, we conclude that $\omega \in \Lambda^2 V$ is decomposable if an only if $\omega \wedge \omega = 0$. Thus, the image of (4-1) is *the Plücker quadric*

$$P \overset{\text{def}}{=} \{\, \omega \in \Lambda^2 V \mid \omega \wedge \omega = 0 \,\} \tag{4-2}$$

If we choose a basis $e_0, e_1, e_2, e_3 \in V$, the monomial basis $e_{ij} = e_i \wedge e_j$ in $\Lambda^2 V$, and write x_{ij} for the homogeneous coordinates along e_{ij}, then straightforward computation

$$\Big(\sum_{i<j} x_{ij} \cdot e_i \wedge e_j\Big) \wedge \Big(\sum_{i<j} x_{ij} \cdot e_i \wedge e_j\Big) = 2\,\big(x_{01}x_{23} - x_{02}x_{13} + x_{03}x_{12}\big) \cdot e_0 \wedge e_1 \wedge e_2 \wedge e_3$$

implies that P is given by the non-degenerated quadratic equation

$$x_{02}x_{13} = x_{01}x_{23} + x_{03}x_{12}\,.$$

Exercise 4.1. Check that the Plücker embedding (4-1) takes a subspace spanned by $u = \sum u_i e_i$, $w = \sum w_j e_j$ to grassmannian quadratic form with coefficients $x_{ij} = u_i w_j - u_j w_i$, i.e. sends a matrix $\begin{pmatrix} u_0 & u_1 & u_2 & u_3 \\ w_0 & w_1 & w_2 & w_3 \end{pmatrix}$ to the collection of its six 2×2-minors $x_{ij} = \det \begin{pmatrix} u_i & u_j \\ w_i & w_j \end{pmatrix}$.

In coordinate-free terms, $P = V(q)$ for the canonical up to a scalar factor quadratic form q on $\Lambda^2 V$ defined by prescription

$$\forall\, \omega_1, \omega_2 \in \Lambda^2 V \quad \omega_1 \wedge \omega_2 = \tilde{q}(\omega_1, \omega_2) \cdot \Omega\,, \tag{4-3}$$

where $\Omega \in \Lambda^4 V \simeq \Bbbk$ is any fixed non-zero vector (unique up to proportionality). Since $\omega_1 \wedge \omega_2 = \omega_2 \wedge \omega_1$ for even grassmannian polynomials, the form $\tilde{q}(\omega_1, \omega_2)$ is symmetric.

Lemma 4.1
$\ell_1 \cap \ell_2 \neq \varnothing$ in $\mathbb{P}_3 \iff \tilde{q}(\mathfrak{u}(\ell_1), \mathfrak{u}(\ell_2)) = \mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell_2) = 0$ in $\mathbb{P}_5$.

Proof. Let $\ell_1 = \mathbb{P}(U_1)$, $\ell_2 = \mathbb{P}(U_2)$. If $U_1 \cap U_2 = 0$, then $V = U_1 \oplus U_2$ and we can choose a basis $e_0, e_1, e_2, e_3 \in V$ such that $\ell_1 = (e_0 e_1)$, $\ell_2 = (e_2 e_3)$. Then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell_2) = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \neq 0$. If $\ell_1 = (ab)$ and $\ell_2 = (ac)$ are intersecting in a, then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell_2) = a \wedge b \wedge a \wedge c = 0$. □

[1] i.e. factorized as exterior product of two vectors
[2] see example 3.5 on p. 61

Corollary 4.1
The Plücker embedding (4-1) is really injective and establishes a bijection between the grassmannian Gr(2, 4) and the Plücker quadric (4-2).

Proof. For any two lines $\ell_1 \neq \ell_2$ on $\mathbb{P}_3$ there exists a third line ℓ which intersect ℓ_1 and does not intersect ℓ_2. Then $\mathfrak{u}(\ell_1) \wedge \mathfrak{u}(\ell) = 0$ and $\mathfrak{u}(\ell_2) \wedge \mathfrak{u}(\ell) \neq 0$ imply $\mathfrak{u}(\ell_1) \neq \mathfrak{u}(\ell_2)$. □

Corollary 4.2
For any point $p = \mathfrak{u}(\ell) \in P$ the intersection $P \cap T_p P$ consists of all $\mathfrak{u}(\ell')$ such that $\ell \cap \ell' \neq \varnothing$.

Proof. This follows from cor. 2.4 on p. 29 and lemma 4.1 above. □

4.1.1 Line nets and line pencils in $\mathbb{P}_3$. A family of lines on $\mathbb{P}_3$ is called *a net of lines* if the Plücker embedding sends it to some plane $\pi \subset P \subset \mathbb{P}_5$. Any plane $\pi \subset P$ is spanned by a triple of non collinear points $p_i = \mathfrak{u}(\ell_i)$, $i = 1, 2, 3$, and lies in the tangent space to each of them. Thus, $\pi \subset P \cap T_{p_1} P \cap T_{p_2} P \cap T_{p_3} P$. By lemma 4.1 and cor. 4.2, the corresponding net of lines consist of all lines that intersect 3 given pairwise intersecting lines $\ell_i \subset \mathbb{P}_3$. There are exactly two geometrically different possibilities fo that:

α-net consists of all lines passing through a given point $O \in \mathbb{P}_3$ and corresponds to *α-plane* $\pi_\alpha(O) \subset P$ spanned by Plücker's images of any 3 non-coplanar lines passing through O

β-net consists of all lines in a given plane $\Pi \in \mathbb{P}_3$ and corresponds to *β-plane* $\pi_\beta(\Pi) \subset P$ spanned by Plücker's images of any 3 lines laying in Π without common intersection.

Any two planes of the same type are always intersecting in precisely one point:

$$\pi_\beta\left(\Pi_1\right) \cap \pi_\beta\left(\Pi_2\right) = \mathfrak{u}\left(\Pi_1 \cap \Pi_2\right), \quad \pi_\alpha\left(O_1\right) \cap \pi_\alpha\left(O_2\right) = \mathfrak{u}\left(\left(O_1 O_2\right)\right).$$

Two planes of different types $\pi_\beta(\Pi)$, $\pi_\alpha(O)$ either do not intersect each other (if $O \notin \Pi$) or are intersecting along a line (if $O \in \Pi$) that corresponds to the pencil of lines $\ell \subset \mathbb{P}_3$ passing through O and laying in Π.

Exercise 4.2. Show that there are no other pencils of lines in $\mathbb{P}_3$, i.e. each line laying on $P \subset \mathbb{P}_5$ has a form $\pi_\beta(\Pi) \cap \pi_\alpha(O)$ for some $O \in \Pi$.

4.1.2 Cell decomposition. Fix some $p \in P$ and a hyperplane $H \simeq \mathbb{P}_3$ complementary to p inside $T_p P \simeq \mathbb{P}_4$. Then intersection $C = P \cap T_p P$ is a simple cone with vertex at p over a smooth quadric $G = H \cap P$ that can be thought of as the Segre quadric in $\mathbb{P}_3 = H$. Choosing some point $p' \in G$ and writing π_α, π_β for planes spanned by p and two lines on G passing through p', wee get a stratification of the Plücker quadric P by closed subvarieties shown on fig. 4◇1 :

$$\begin{tikzcd} & & \pi_\alpha \arrow[dr, hook] & & \\ p \arrow[r, hook] & \pi_\alpha \cap \pi_\beta \arrow[ur, hook] \arrow[dr, hook] & & C \arrow[r, hook] & P \\ & & \pi_\beta \arrow[ur, hook] & & \end{tikzcd} \tag{4-4}$$

It decomposes P as disjoint union of open cells[1] isomorphic to affine spaces:

$$\mathrm{Gr}(2,4) \;=\; \mathbb{A}^0 \sqcup \mathbb{A}^1 \sqcup \begin{pmatrix} \mathbb{A}^2 \\ \sqcup \\ \mathbb{A}^2 \end{pmatrix} \sqcup \mathbb{A}^3 \sqcup \mathbb{A}^4$$

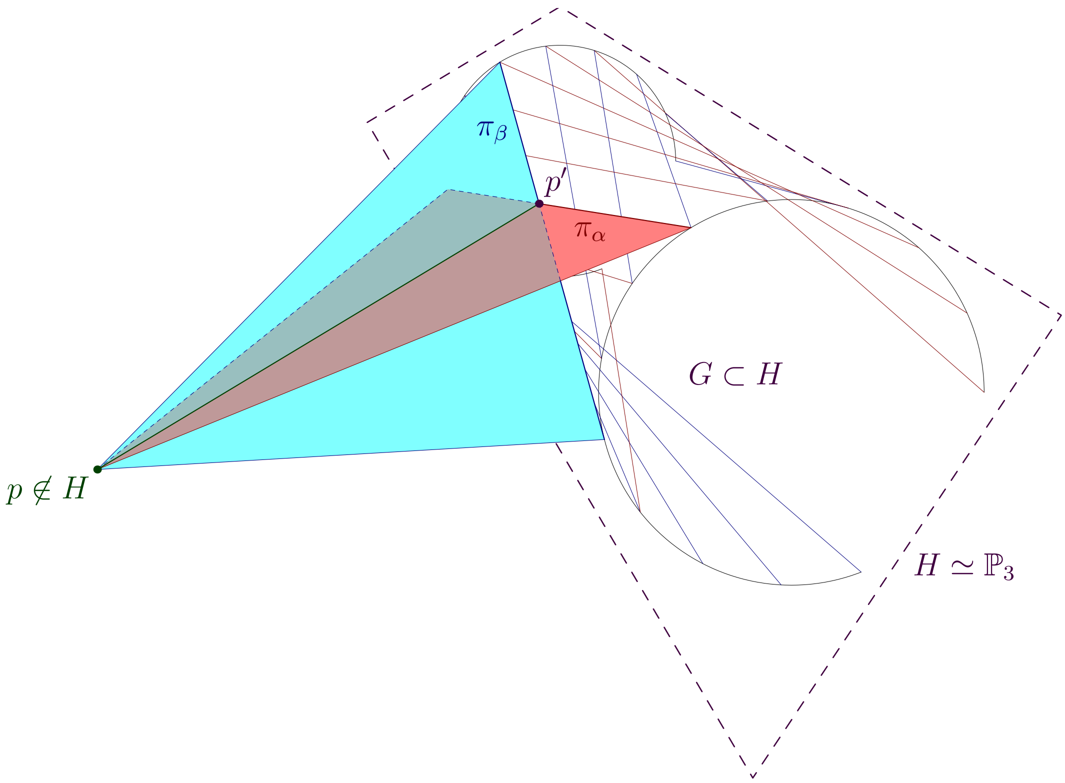


Рис. 4◇1. The cone $C = P \cap T_p P \subset \mathbb{P}_4 = T_p P$.

It starts with $\{p\} \simeq \mathbb{A}^0$, then stays projective line without this point: $(\pi_\alpha \cap \pi_\beta) \smallsetminus p \simeq \mathbb{A}^1$, then the pair of projective spaces without this line: $\pi_\alpha \smallsetminus (\pi_\alpha \cap \pi_\beta) \simeq \pi_\beta \smallsetminus (\pi_\alpha \cap \pi_\beta) \simeq \mathbb{A}^2$, then we have the cone C over G without these two planes: $C \smallsetminus (\pi_\alpha \cup \pi_\beta) \simeq \mathbb{A}^1 \times \big(G \smallsetminus (G \cap T_{p'}G)\big)$. Finally, there are natural identifications $G \smallsetminus (G \cap T_{p'}G) \simeq \mathbb{A}^2$ and $Q \smallsetminus C \simeq \mathbb{A}^4$ provided by the next lemma.

Lemma 4.2
Projection $p : Q \to \Pi$, of a smooth quadric $Q \subset \mathbb{P}_n$ from a point $p \in Q$ onto hyperplane $\Pi \not\ni p$, establishes a bijection between $Q \smallsetminus T_p Q$ and $\mathbb{A}^{n-1} = \Pi \smallsetminus T_p Q$.

Proof. Each non-tangent line passing through p does intersect Q in precisely one point[2] different from p. All these lines stay in bijections with the points of $\mathbb{A}^{n-1} = \Pi \smallsetminus T_p Q$. □

[1] each affine cell of this decomposition is an open dense subset of the corresponding stratum in (4-4) complementary to the union of all the strata of lower dimension contained in the stratum we deal with

[2] if we write x for this point and put $y = (px) \cap \Pi$, then it follows from Vieta formulas that x and y are rational functions of each other; thus, the bijection of lemma is actually *an isomorphism of affine algebraic varieties*

Exercise 4.3. If you have some experience in CW-topology, show that integer homologies of complex grassmannian are

$$H_m\left(\mathrm{Gr}(2,\mathbb{C}^4),\ \mathbb{Z}\right) = \begin{cases} 0 & \text{for odd } m \text{ and } m > 8 \\ \mathbb{Z} & \text{for } m = 0,\ 2,\ 6,\ 8 \\ \mathbb{Z}\oplus\mathbb{Z} & \text{for } m = 4 \end{cases}$$

Try to compute integer homologies $H_m\left(\mathrm{Gr}(2,\mathbb{R}^4),\ \mathbb{Z}\right)$ of the real grassmannian (the boundary maps are non trivial here).

4.2 Lagrangian grassmannian LGr(2, 4) and lines on a smooth quadric in $\mathbb{P}_4$. Let us equip 4-dimensional vector space V with non-degenerated skew-symmetric bilinear form Ω and fix some non-zero vector $\delta \in \Lambda^4 V$. Then there exists a unique grassmannian quadratic form $\omega \in \Lambda^2 V$ satisfying

$$\forall\, a, b \in V \quad \omega \wedge a \wedge b = \Omega(a,b)\cdot\delta\,. \tag{4-5}$$

Write $W = \operatorname{Ann}\hat{q}(\omega)$ for orthogonal complement to ω w.r.t. the Plücker quadratic form q on $\Lambda^2 V$ defined in formula (4-3), p. 74. Then $Z = \mathbb{P}(W) \simeq \mathbb{P}_4$ is the polar hyperplane of ω w.r.t. the Plücker quadric $P \in \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$.

Exercise 4.4. Check that $\omega \notin P$.

Since $\omega \notin P$ the intersection $R \stackrel{\text{def}}{=} Z \cap P$ is a smooth quadric in $Z \simeq \mathbb{P}_4$. Its points stay in bijection with *lagrangian subspaces*[1] in V w.r.t. symplectic form Ω, because of (4-5) and (4-3) which say together that line $(ab) \subset \mathbb{P}_3$ has $\Omega(a,b) = 0$ iff $\tilde{q}(\omega,\ a\wedge b) = 0$.

By this reason, R is called *lagrangian grassmannian* of symplectic form Ω and is usually denoted by $\mathrm{LGr}(2,4) = \mathrm{LGr}(\Omega, V)$.

It follows from general theory developed in (n° 2.5) that R does not contain planes but is filled by lines in such a way that lines passing through a given point $r \in R$ rule a simple cone over a smooth conic with the vertex at r. The variety of lines laying in a given algebraic manifold X is called *the Fano Variety of* X and is denoted by $F(X)$.

Proposition 4.1
There is well defined isomorphism $\mathbb{P}_3 = \mathbb{P}(V) \xrightarrow{\sim} F(R)$ sending a point $p \in \mathbb{P}(V)$ to the pencil of all lagrangian lines in $\mathbb{P}(V)$ passing through p.

Proof. We have seen in n° 4.1.1 (esp. exrs. 4.2 on p. 75) that each line $L \subset P$ is an intersection of α- and β-planes: $L = \pi_p \cap \pi(\Pi)$, i.e. consists of all lines passing through some point $p \in \mathbb{P}(V)$ and laying in some plane Π. If $L \subset R = P \cap Z$, then L consists of all lagrangian lines passing trough p and laying in Π. On the other hand, a line (px), which passes through a given point $p \in \mathbb{P}(V)$, is lagrangian iff $\Omega(p,x) = 0$. Thus, all lagrangian lines passing through an arbitrary point p lie a plane that is orthogonal to p w.r.t. symplectic form Ω, that is, form a pencil. □

[1] n-dimensional subspace U in $2n$-dimensional subspace V equipped with non-degenerated skew-symmetric bilinear form Ω is called *lagrangian* if $\Omega(u,w) = 0$ for all $u, w \in U$; thus lagrangian subspaces are skew-symmetric analogues of maximal isotropic subspaces of non-degenerated symmetric forms

4.3 Grassmannians Gr(k,n). A variety of all of all m-dimensional vector subspaces in a given d-dimensional vector space V is called *a grassmannian variety* and is denoted by $\mathrm{Gr}(m,d)$ or by $\mathrm{Gr}(m,V)$, when the nature of V is essential. In projective world, grassmannian $\mathrm{Gr}(m,d)$ parametrizes $(m-1)$-dimensional projective subspaces in $\mathbb{P}_{d-1}$. Simplest grassmannians are the projective spaces $\mathbb{P}_n = \mathbb{P}(V) = \mathrm{Gr}(1,V) = \mathrm{Gr}(1,n+1)$ and $\mathbb{P}_n^\times = \mathbb{P}(V^*) = \mathrm{Gr}(n,V) = \mathrm{Gr}(n,n+1)$. More generally, the duality $U \leftrightarrow \mathrm{Ann}\, U$ establishes canonical bijection $\mathrm{Gr}(m,V) \leftrightarrow \mathrm{Gr}(d-m,V^*)$, where $d = \dim V$. Simplest grassmannian besides the projective spaces is $\mathrm{Gr}(2,4)$ considered in the previous section.

Exercise 4.5. Let $\dim V = 4$ and $\hat{q} : V \xrightarrow{\sim} V^*$ be a correlation provided by some smooth quadric $Q = V(q) \subset \mathbb{P}(V)$. Show that prescription $U \mapsto \mathrm{Ann}\, \hat{q}(U)$ defines an automorphism $\mathrm{Gr}(2,V) \xrightarrow{\sim} \mathrm{Gr}(2,V)$ that sends α-planes on $\mathrm{Gr}(2,4)$ to the β-planes and vice versa.

.

4.3.1 The Plücker embedding takes m-dimensional subspace $U \subset V$ to the 1-dimensional subspace $\Lambda^m U \subset \Lambda^m V$. This gives the mapping

$$\mathfrak{u} : \mathrm{Gr}(m,V) \hookrightarrow \mathbb{P}(\Lambda^m V)\,. \tag{4-6}$$

If vectors $u_1, u_2, \dots, u_m$ form a basis in subspace $U \subset V$, then $\mathfrak{u}(U) = \Bbbk \cdot u_1 \wedge u_2 \wedge \cdots \wedge u_m$. Since for any two distinct m-dimensional subspaces $U \neq W$ we can choose a basis

$$v_1, v_2, \dots, v_r,\ u_1, u_2, \dots, u_{m-r},\ w_1, w_2, \dots, w_{m-r},\ v_{2m-r},\ v_{2m-r+1},\ \dots\ ,\ v_n \in V$$

such that $v_1, v_2, \dots, v_r$ form a basis of $U \cap W$ and $u_1, u_2, \dots, u_{m-r}$, $w_1, w_2, \dots, w_{m-r}$ complete it to some bases of U , W, the mapping (4-6) sends U and V to distinct basic monomials

$$v_1 \wedge \cdots \wedge v_r \wedge u_1 \wedge \cdots \wedge u_{m-r} \neq v_1 \wedge \cdots \wedge v_r \wedge w_1 \wedge \cdots \wedge w_{m-r}$$

of $\Lambda^m V$. Thus, the Plücker embedding (4-6) is actually injective and establishes bijection between $\mathrm{Gr}(m,d)$ and the variety of decomposable grassmannian polynomials $\omega \in \Lambda^m V$. By prop. 3.4 on p. 70, the latter is described as an intersection of quadrics provided by the Plücker relations from the statement (3) of prop. 3.4.

Algebraically, the grassmannian variety $\mathrm{Gr}(m,d) \subset \mathbb{P}(\Lambda^m V)$ is a straightforward skew-commutative analogue of the Veronese variety $\mathcal{V}(m,d) \subset \mathbb{P}(S^m V)$: both consist of non-zero homogeneous degree m polynomials of *the maximal degeneracy*, that is, having the minimal possible non-zero linear support. For ordinary commutative polynomial $f \in S^m V$ this means that $\dim \mathrm{Supp}\, f = 1$ and $f = v^n$ for some $v \in V$. For non-zero grassmannian polynomial $\omega \in \Lambda^m V$ the minimal $\dim \mathrm{Supp}\, \omega = m$ and in this case $\omega = w_1 \wedge w_2 \wedge \cdots \wedge w_m$ for some $w_1, w_2, \dots, w_m \in V$.

4.3.2 Matrix notations and the Plücker coordinates. As soon a basis $e_1, e_2, \dots, e_d \in V$ is chosen, one can represent a point $U \in \mathrm{Gr}(m,d)$ by an equivalence class of $m \times d$-matrix $A(U)$ whose rows are the coordinates of vectors $u_1, u_2, \dots, u_m \subset U$ that form a basis in U. Another choice of basis in U changes $A(U)$ via left multiplication by a matrix $C \in \mathrm{GL}_m(\Bbbk)$. Thus, the points of grassmannian $\mathrm{Gr}(m,d)$ are the left $\mathrm{GL}_m(\Bbbk)$-orbits in $\mathrm{Mat}_{m\times d}(\Bbbk)$. These agrees with homogeneous coordinates on $\mathbb{P}(V) = \mathrm{Gr}(1,V)$, which are the rows (i.e. $1 \times d$-matrices) up to multiplication by elements of $\mathrm{GL}_1(\Bbbk) = \Bbbk^*$.

Choosing the standard monomial basis $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$ in $\Lambda^m V$, one can describe the Plücker embedding (4-6) as a map that takes matrix $A(U)$ to a point whose coordinate x_I in the basis e_I equals the maximal minor $a_I(U)$ of $A(U)$ situated in columns $i_1, i_2, \dots, i_m$. Indeed,

$$u_1 \wedge u_2 \wedge \cdots \wedge u_m = \Big(\sum_{i_1} a_{1i_1} e_{i_1}\Big) \wedge \Big(\sum_{i_2} a_{2i_2} e_{i_2}\Big) \wedge \cdots \wedge \Big(\sum_{i_m} a_{mi_n} e_{i_m}\Big) =$$
$$= \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant m} \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) \alpha_{1i_{\sigma(1)}} a_{2i_{\sigma(2)}} \cdots a_{mi_{\sigma(m)}} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} = \sum_I a_I e_I$$

where $a_I = \det\left(a_{i, i_j}\right)$. Maximal minors $a_I = a_I(U)$ of $A(U)$ are called *the Plücker coordinates* of U. It follows from example 3.3 that the left multiplication $A(U) \mapsto C \cdot A(U)$ by some $C \in \mathrm{GL}_m(\Bbbk)$ multiplies all the Plücker coordinates $a_I(U)$ by $\det C$.

4.3.3 Affine charts and affine coordinates. The standard affine cards $U_{x_I} \subset \mathbb{P}(\Lambda^m V)$, where I runs through increasing collections $i_1, i_2, \dots, i_m$, cover the grassmannian $\mathrm{Gr}(m, V)$. Let us write $\mathcal{U}_I \subset \mathrm{Gr}(m, V)$ for the intersection $U_{x_I} \cap \mathrm{Gr}(m, V)$ and call it *a standard affine chart* on $\mathrm{Gr}(m, V)$. This chart consists of all $U \subset V$ such that $A(U)$ has non zero maximal minor in the columns $i_1, i_2, \dots, i_m$.

Geometrically, this means that $U \subset V$ is isomorphically mapped onto coordinate subspace of V spanned by the basic vectors e_i with $i \in I$ by the projection along the complementary coordinate subspace spanned by e_j with $j \notin I$. In particular, there exist a unique basis of U that consists of the preimages of basic vectors e_i, $i \in I$, under this projection.

Algebraically, this means that each $U \in \mathcal{U}_I$ has *a unique* matrix representation $A^{(I)} = A^{(I)}(U)$ such that $m \times m$-submatrix $A^{(I)}_I \subset A^{(I)}$ situated in I-columns is *the identity* $m \times m$ *matrix*. This matrix representation is obtained from an arbitrary representation $A = A(U)$ as $A^{(I)} = A_I^{-1} \cdot A$, where $A_I \subset A$ is $m \times m$-submatrix formed by I-columns.

Thus, the points of the standard chart $\mathfrak{U}_I \subset \mathrm{Gr}(m, d)$ stay in bijection with $m \times d$ matrices $A^{(I)}$ with E in I-columns and can be identified with the affine space $\mathbb{A}^{m(d-m)}$ coordinated by the matrix elements $\left(a^{(I)}_{\mu\nu}\right)$ of $A^{(I)}$ staying outside the columns $(i_1, i_2, \dots, i_m)$. In particular,

$$\dim \mathrm{Gr}(m, d) = m(d - m)\,.$$

Exercise 4.6. If you had deal with differential, analytic, or algebraic geometry, check that real (resp. complex, or arbitrary) grassmannians are smooth (resp. holomorphic, or algebraic) manifolds.

4.4 Cell decomposition. The Gauss method shows that each $U \subset V$ admits *a unique* basis $u_1, u_2, \dots, u_m \in U$ whose matrix A is *a reduced step matrix*. By the definition, this means that there is the identity $m \times m$-submatrix $E \subset A$ situated in some columns $j_1, j_2, \dots, j_m$ and each row of A vanishes at the left of the unity coming from this identity submatrix, that is, for each $i = 1, 2, \dots, m$ and any $j < j_i \quad a_{ij} = 0$.

Exercise 4.7. Make it sure that the rows of distinct reduced step matrices span distinct subspaces in $\Bbbk^d$.

Thus, there exist a bijection between $\mathrm{Gr}(m, d)$ and strong step $(m \times d)$-matrices of rank m. The latter split into disjoint union of affine spaces. Namely, all strong step matrices of *prescribed shape*, i.e. containing the identity submatrix in prescribed columns $i_1, i_2, \dots, i_m$, have exactly

$$md - m^2 - (i_1 - 1) - (i_2 - 2) - \cdots - (i_m - m) = \dim \mathrm{Gr}(m, d) - \sum_{\nu=1}^{m} (i_\nu - \nu)$$

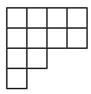
free entries to put there any constants from $\Bbbk$. Hence, $\mathrm{Gr}(m,d)$ is a disjoint union of $\binom{d}{m}$ affine cells $\mathfrak{A}_I$ enumerated by increasing collections $I=(i_1,i_2,\dots,i_m)\subset(1,2,\ \dots\ ,d)$. The I-th cell $\mathfrak{A}_I$ is isomorphic to affine space and has codimension $\sum\limits_{\nu=1}^{m}(i_\nu-\nu)=|I|-\frac{m(m+1)}{2}$ in $\mathrm{Gr}(m,d)$.

4.4.1 Young diagram notations. Another common way of numbering the disjoint affine cells $\mathfrak{A}_I\subset\mathrm{Gr}(m,d)$ replaces increasing subset $I=i_1,i_2,\dots,i_m$ by *a partition* $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_m)$ that is non-decreasing collection of non-negative integers

$$\lambda_1\geqslant\lambda_2\geqslant\ \cdots\ \geqslant\lambda_m\geqslant 0$$

whose ν th from the right element $\lambda_{m+1-\nu}$ is equal to ν th from the left difference $(i_\nu-\nu)$ in the increasing collection $i_1,i_2,\dots,i_n$. Thus, the identity $m\times m$-submatrix of reduced step matrix of type $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_m)$ is situated in columns $i_\nu=(m+1-\nu)+\lambda_{m+1-\nu}$. In other words, λ_ν equals the difference that appears in the ν th row from the bottom between the *actual* position of the leftmost non-zero element and the leftmost *possible* position for such element.

Partitions are visualised by means of *Young diagrams*, that is, aligned to the left cellular strips of lengths $\lambda_1,\lambda_2,\dots,\lambda_m$. Total number of cells $|\lambda|\overset{\text{def}}{=}\sum\lambda_i$ is called *a weight*[1] of the diagram. A number of rows $\ell(\lambda)\overset{\text{def}}{=}\max(k\mid\lambda_k>0)$ is called *a length* of the diagram. For example, partition

$(4,4,2,1)\ \ \leftrightsquigarrow$ 

has length 4 and weight 11. In grassmannian $\mathrm{Gr}(4,10)$ it defines 13-dimensional affine cell formed by the subspaces representable by reduced step matrices of shape

$$\begin{pmatrix} 0 & 1 & * & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{pmatrix}.$$

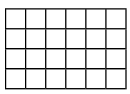
The differences between actual leftmost units (colored in blue) and the leftmost possible positions for them (colored in red) listen from the bottom to the top are 4, 4, 2, 1, i.e. coincide with the partition.

The zero partition $(0,0,0,0)$ corresponds to the leftmost possible positions of steps and defines 24-dimensional affine cell of spaces representable by matrices of the shape

$$\begin{pmatrix} 1 & 0 & 0 & 0 & * & * & * & * & * & * \\ 0 & 1 & 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 1 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \end{pmatrix}$$

i.e. the standard affine chart $\mathfrak{U}_{(1,2,3,4)}$ of the grassmannian $\mathrm{Gr}(4,10)$.

Maximal possible partition $(6,6,6,6)$ whose Young diagram exhausted the whole of rectangle



[1]Young diagrams of a given weight n stay in bijection with partitions of n into a sum of non-ordered positive integers (this explains the terminology)

corresponds to the zero-dimensional one point cell represented by the reduced step matrix of the rightmost possible shape

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 4.8. Make it sure that collections of increasing indexes $I \subset (1, 2, \dots, d)$ stay in bijections with Young diagrams λ contained in rectangle of size $m \times (d - m)$.

Thus, grassmannian $\mathrm{Gr}(m, d) = \bigsqcup_{\lambda} \sigma^\circ_\lambda$ is disjoint union of affine spaces

$$\sigma^\circ_\lambda = \mathbb{A}^{m(d-m)-|\lambda|}$$

called *open Schubert cells* and numbered by the Young diagrams λ contained in the rectangle of size $m \times (d - m)$. The closure of σ°_λ in $\mathrm{Gr}(m, d)$ is called (*closed*) *Schubert cycle*.

Example 4.1 (homologies of complex grassmannians)
Closed Shubert cycles σ_λ form a free basis of abelian group of integer homologies of complex grassmannian

$$\Lambda(m, d) \overset{\text{def}}{=} H_* \left(\mathrm{Gr}(m, \mathbb{C}^d), \mathbb{Z}\right) ,$$

because the cell decomposition just constructed does not contain cells of odd real dimension. The latter means that all boundary operators in the cell chain complex vanish.

Exercise 4.9*. List all open cells σ°_μ containing in the closure of a given open cell σ°_l and try to evaluate the boundary operators in the cell chain complex of real grassmannian $\mathrm{Gr}\left(m, \mathbb{R}^d\right)$.

4.4.2 Schubert calculus. Topological intersection of cycles provides abelian group $\Lambda(m, d)$ with a structure of commutative ring. It turns to be a truncated ring of symmetric functions:

$$\Lambda(m, d) \simeq \mathbb{Z}[\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m]/(\eta_{d-m+1}, \dots, \eta_{d-1}, \eta_d) \tag{4-7}$$

where ε_k and η_k stay for *elementary*[1] and *complete*[2] symmetric polynomials of degree k in m auxiliary variables $x_1, x_2, \dots, x_m$. By the main theorem about symmetric functions, all η_k are uniquely expanded as polynomials in $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$ and factorisation in (4-7) is modulo the ideal spanned by those polynomials expanding $\eta_{d-m+1}, \dots, \eta_d$ trough ε_i's.

The isomorphism (4-7) sends a Schubert cycle σ_λ to the Schur polynomial s_λ, that is the sum of all monomials in $x_1, x_2, \dots, x_m$ obtained as follows: fill the cells of diagram λ by letters $x_1, x_2, \dots, x_m$ (each letter can be used any number of times) in such a way that the indexes of variables strictly increase along the columns from the top to the bottom and non-strictly increase along the rows from the left to the right, then multiply all the entries into a monomial of degree $|\lambda|$. For example, $s_{(k)} = \eta_k$, where (k) means one row of length k, and $s_{(1^k)} = \varepsilon_k$, where $\left(1^k\right) = (1, \dots, 1)$ means one column of height k.

All proofs of the isomorphism (4-7) that I know are non direct. Anyway, they use quite sophisticated combinatorics of symmetric functions besides the geometry of grassmannians. The geometric part of the proof establishes two basic intersection rules:

[1]sum of all k-*linear* monomials of degree k

[2]sum of all degree k monomials in $x_1, x_2, \dots, x_m$ at all

1) An intersection of cycles σ_λ, σ_μ of complementary codimensions $|\lambda| + |\mu| = m(d+m)$ is non empty iff their diagrams are complementary[1] and in this case the intersection consists of one point corresponding to the matrix of shape λ whose $*$-entries equal zero.

2) *The Pieri rules*: for any diagram λ we have $\sigma_\lambda \sigma_{(k)} = \sum \sigma_\mu$ and $\sigma_\lambda \sigma_{(1^k)} = \sum \sigma_\nu$, where μ, ν run through Young diagrams obtained by adding k cells to λ in such a way that all added cells stay in distinct rows in μ and stay in distinct columns in ν.

The details can be found in *P. Griffits, J. Harris. Principles of Algebraic Geometry, I.*

Combinatorial part of the proof verifies that the Schur polynomials satisfy the Pieri rules and form a basis of the $\mathbb{Z}$-module Λ of symmetric polynomials in countable number of variables with integer coefficients. Direct computation shows that the Pieri rules uniquely determine the multiplicative structure on Λ. This leads to surjective homomorphism $\Lambda \twoheadrightarrow \Lambda(m,d)$ sending $s_\lambda \mapsto \sigma_\lambda$. Its kernel is uniquely determined as soon we know that the classes of s_λ with $\lambda \subseteq (d^m)$ form a basis in the factor ring and satisfy there the first intersection rule mentioned above. The details can be found in two remarkable *W. Fulton*'s books *Young Tableaux* and *Intersection Theory*.

Avoiding general theory, we compute the intersection ring of $\mathrm{Gr}(2,4)$ in example 4.2.

Exercise 4.10. Verify that in the notations of n° 4.1.1 six Schubert cycles on the Plücker quadric $P = \mathrm{Gr}(2,4) \subset \mathbb{P}_5$ are: $\sigma_{00} = P$; $\sigma_{22} = p = (0 : 0 : 0 : 0 : 0 : 1) \in \mathbb{P}_5$; $\sigma_{10} = P \cap T_p P$; $\sigma_{11} = \pi_\alpha(O)$, where $O = (0 : 0 : 0 : 1) \in \mathbb{P}_3$; $\sigma_{20} = \pi_\beta(\Pi)$, where $\Pi \subset \mathbb{P}_3$ is given by linear equation $x_0 = 0$; $\sigma_{21} = \pi_\alpha(O) \cap \pi_\beta(\Pi)$.

Example 4.2 (intersection theory on $\mathrm{Gr}(2,4)$)
If $\mathrm{codim}\sigma_\lambda + \mathrm{codim}\sigma_\mu < 4$, then $\sigma_\lambda\sigma_\mu = \varnothing$, certainly. Intersections of cycles of complementary codimensions was described actually in n° 4.1.1 and exrs. 4.2: $\sigma_{10}\sigma_{21} = \sigma_{20}^2 = \sigma_{11}^2 = \sigma_{22}$ and $\sigma_{20}\sigma_{11} = 0$. By the same geometric reasons $\sigma_{10}\sigma_{20} = \sigma_{10}\sigma_{11} = \sigma_{21}$. In order to compute σ_{10}^2, let us realise σ_{10} as $\sigma_{10}(\ell) = P \cap T_{\mathfrak{u}(\ell)} P = \{\ell'' \subset \mathbb{P}_3 \mid \ell \cap \ell'' \neq \varnothing\}$. Then σ_{10}^2 is a homology class of an intersection $\sigma_{10}(\ell) \cap \sigma_{10}(\ell')$ that generically is represented by the Segre quadric shown in fig. 4◇1 on p. 76. Let us move ℓ' to a position where $\ell' \cap \ell \neq \varnothing$ but $\ell' \neq \ell$. Under this moving the Segre quadric is deformed inside its homology class to the union of two planes: α-net with the centre at $O = \ell \cap \ell'$ and β-net in the plane Π spanned by $\ell \cup \ell'$, i.e. we get $\sigma_{10}(\ell) \cap \sigma_{10}(\ell') = \pi_\alpha(O) \cup \pi_\beta(\Pi)$. Thus, $\sigma_{10}^2 = \sigma_{20} + \sigma_{11}$.

This leads to «topological» solution of exrs. 2.10 and prb. 2.10 asked how many lines does intersect 4 given mutually skew lines in $\mathbb{P}_3$. If 4 given lines are general enough[2], then formal computation in $\Lambda(2,4)$: $\sigma_{10}^4 = (\sigma_{20} + \sigma_{11})^2 = \sigma_{20}^2 + \sigma_{11}^2 = 2\sigma_{22}$ tells us that there are 2 such lines in general.

[1] i.e. can be fitted together without holes and overlaps to assemble $m \times (d-m)$ rectangle

[2] namely, the intersection of cycles $\sigma_{10}(\ell_i)$ provided by these lines represents the topological 4-fold self-intersectionσ_{10}^4

Home task problems to §4

Problem 4.1. Is there a complex 2×4-matrix whose set of 2×2-minors is a) $\{2,\ 3,\ 4,\ 5,\ 6,\ 7\}$ b) $\{3,\ 4,\ 5,\ 6,\ 7,\ 8\}$? If so, give an explicit example of such a matrix.

Problem 4.2. Let $G = V(g) \subset \mathbb{P}_3 = \mathbb{P}(V)$ be a smooth quadric. Define a bilinear form $\Lambda^2\widetilde{g}$ on $\Lambda^2 V$ by prescription $\Lambda^2\widetilde{g}(\,v_1 \wedge v_2\,,\, w_1 \wedge w_2\,) \overset{\text{def}}{=} \det \begin{pmatrix} \widetilde{g}(v_1, w_1) & \widetilde{g}(v_1, w_2) \\ \widetilde{g}(v_2, w_1) & \widetilde{g}(v_2, w_2) \end{pmatrix}$. Check that it is symmetric and non-degenerated and write down its Gram matrix in a standard monomial basis of $\Lambda^2 V$ built of some g-orthonormal basis in V.

Problem 4.3. In continuation of prb. 4.2 take $g(A) = \det A$ as the quadratic form on the space $V = \operatorname{Hom}(U_-, U_+)$, where $U_\pm \simeq \mathbb{C}^2$, write $\Lambda^2 g$ for the smooth quadratic form on $\Lambda^2 V$ that takes $v_1 \wedge v_2$ to the Gram determinant $\Lambda^2 g\left(v_1 \wedge v_2\right) \overset{\text{def}}{=} \det \begin{pmatrix} \widetilde{g}(v_1, v_1) & \widetilde{g}(v_1, v_2) \\ \widetilde{g}(v_2, v_1) & \widetilde{g}(v_2, v_2) \end{pmatrix}$, and write $P = \{\omega \in \Lambda^2 V \mid \omega \wedge \omega = 0\} \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$ for the Plücker quadric. Show that

a) the intersection of quadrics $V(\Lambda^2 g) \cap P \subset \mathbb{P}_5$ consists of all lines in $\mathbb{P}_3 = \mathbb{P}(V)$ tangent to the Segre quadric $G = V(g) \subset \mathbb{P}_3$.

b) the Plücker embedding $\operatorname{Gr}(2, V) \xrightarrow{\sim} P \subset \mathbb{P}(\Lambda^2 V)$ sends two line rulings of the Segre quadric G to a pair of distinct smooth conics $C_\pm \subset P$ that are cut out of the Plücker quadric by a pair of complementary planes $\Lambda_- = \mathbb{P}\left(S^2 U_-^* \otimes \Lambda^2 U_+\right)$ and $\Lambda_+ = \mathbb{P}\left(\Lambda^2 U_-^* \otimes S^2 U_+\right)$ embedded into $\mathbb{P}_5 = \mathbb{P}(\Lambda^2 \operatorname{Hom}(U_-, U_+))$ by means of decomposition (3-55) from prb. 3.6 on p. 72.

c) both conics $C_- \subset \mathbb{P}\left(S^2 U_-^* \otimes \Lambda^2 U_+\right)$ and $C_+ \subset \mathbb{P}\left(\Lambda^2 U_-^* \otimes S^2 U_+\right)$ are the images of the Veronese embeddings $\mathbb{P}(U_-^*) \subset \mathbb{P}\left(S^2 U_-^*\right)$ and $\mathbb{P}(U_+) \subset \mathbb{P}\left(S^2 U_+\right)$, i.e. we have the following commutative diagram of the Plücker – Segre – Veronese interactions[1]:

$$\begin{tikzcd}
\mathbb{P}(U_+) \arrow[rrr, hook, "\text{Veronese}"'] & & & \mathbb{P}(S^2 U_+) \simeq \Lambda_+ \arrow[d, hook] \\
\mathbb{P}_1^+ \times \mathbb{P}_1^- \arrow[u, "\pi_+"] \arrow[d, "\pi_-"'] \arrow[r, "\text{Segre}", "\sim"'] & G \subset \mathbb{P}\operatorname{Hom}(U_-, U_+) \arrow[rr, dashed, "\text{Plücker}"] & & P \subset \mathbb{P}\begin{pmatrix} \Lambda^2 U_-^* \otimes S^2 U_+ \\ \oplus \\ S^2 U_-^* \otimes \Lambda^2 U_+ \end{pmatrix} \\
\mathbb{P}(U_-^*) \arrow[rrr, hook, "\text{Veronese}"] & & & \mathbb{P}(S^2 U_-^*) \simeq \Lambda_- \arrow[u, hook]
\end{tikzcd}$$

d) (Hodge's star) Associated with smooth quadretic form g on V is *the Hodge star-operator*

$$* : \Lambda^2 V \xrightarrow{\omega \mapsto \omega^*} \Lambda^2 V\,,$$

defined by prescription $\forall\, \omega_1, \omega_2 \in \Lambda^2 V \quad \omega_1 \wedge \omega_2^* = \Lambda^2\widetilde{g}(\omega_1, \omega_2) \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4$, where $e_1,\ e_2,\ e_3,\ e_4 \in V$ is an orthonormal basis for g. Verify that this definition does not depend on a choice of orthonormal basis, find eigenvalues and eigenspaces of $*$, and show their place in the previous picture.

[1]Plücker is dashed, because it takes lines to points

§5 Real affine convex geometry

On default, everywhere in §5 except for very beginning of n° 5.1 we deal with finite dimensional vector spaces over the field of real numbers $\mathbb{R}$ and with affine spaces associated with them.

5.1 Linear affine geometry. We say that a set A is *an affine space* over a vector space V, if for each $v \in V$ a *parallel displacement* operator[1] $\tau_v : A \to A$ is given such that

$$1)\ \tau_0 = \mathrm{Id}_A\,, \qquad 2)\ \forall\, v, w \in V \quad \tau_u \circ \tau_w = \tau_{u+w}\,, \tag{5-1}$$

$$3)\ \forall\, p, q \in A\ \exists \text{ unique } v \in V\ :\ \tau_v(p) = q\,. \tag{5-2}$$

For $a \in A$ and $v \in V$ we often write $a + v$ instead of $\tau_v(a)$. Conditions (5-1) mean that additive group of V acts on A and this action is simply transitive: any $q \in A$ is obtained from any $p \in A$ by a unique shift operator τ_v. Vector $v \in V$ that produces this shift is denoted by $\overrightarrow{pq}$. Thus, $q = p + \overrightarrow{pq}$. Also it follows from (5-1) that $\overrightarrow{pp} = 0$ and $\forall\, p, q, r \in A$ $\overrightarrow{pq} + \overrightarrow{qr} = \overrightarrow{pr}$.

Exercise 5.1. Make it clear that $\overrightarrow{pq} = -\overrightarrow{qp}$ and $\overrightarrow{pq} = \overrightarrow{rs} \iff \overrightarrow{ps} = \overrightarrow{qr}$.

The affinization $\mathbb{A}(V)$ discussed in n° 1.4 on p. 11 is a particular example of affine space. In $\mathbb{A}(V)$ the shift operator τ_v takes $u \mapsto u + v$.

Exercise 5.2. Show that the set of monic[2] polynomials of degree n in one variable is an affine space over the vector space of all polynomials of degree $\leqslant (n-1)$.

It follows from (5-2) that a choice of a point $o \in A$ establishes a bijection between A and V by sending a point $p \in A$ to the vector $\overrightarrow{op} \in V$. This bijection is called *a vectorization* of A with the origin at $o \in A$.

5.1.1 Barycentric combinations. For any collections of points $p_1, p_2, \dots, p_m \in A$ and constants $\lambda_1, \lambda_2, \dots, \lambda_m \in \Bbbk$ consider a point

$$\sum_i \lambda_i p_i \stackrel{\text{def}}{=} o + \sum_i \lambda_i \cdot \overrightarrow{op}_i\,, \tag{5-3}$$

where $o \in A$ is an arbitrarily chosen point. We claim that R.H.S. of (5-3) does not depend on a choice of o iff $\sum \lambda_i = 1$. Indeed, taking another point r instead of o we get in R.H.S. of (5-3) $r + \sum \lambda_i \cdot \overrightarrow{rp}_i$. Subtracting this from R.H.S. of (5-3), we get

$$\overrightarrow{ro} + \sum_i \lambda_i \cdot \left(\overrightarrow{op}_i - \overrightarrow{rp}_i\right) = \overrightarrow{ro} + \left(\sum_i \lambda_i\right) \cdot \overrightarrow{or} = \left(1 - \sum_i \lambda_i\right) \cdot \overrightarrow{ro}\,,$$

which vanishes iff $\sum \lambda_i = 1$ as soon as $r \neq o$.

Definition 5.1
If $\sum \lambda_i = 1$, then the point $\sum \lambda_i p_i \in A$ defined by equation (5-3) is called *a barycentric combination* of points $p_1, p_2, \dots, p_m$ with weights $\lambda_1, \lambda_2, \dots, \lambda_m$.

[1] also called *a shift*

[2] recall that *monic* means «with leading coefficient 1»

5.1.2 Projective viewpoint. Consider a vector space W^* of dimension $n+1$, pick up non-zero covector $\xi \in W^*$ and put $V = \operatorname{Ann}\xi \subset W$. Then a convenient model of affine space $\mathbb{A}^n$ associated with V is affine chart $U_\xi = \{w \in W \mid \langle \xi , w \rangle = 1\}$ in the projective space $\mathbb{P}_n = \mathbb{P}(W)$. A linear combination $\sum \lambda_i p_i$, of vectors $p_i \in U_\xi$, lies in U_ξ iff $\left\langle \xi , \sum \lambda_i p_i \right\rangle = \sum \lambda_i = 1$. Thus, the barycentric combinations of points in U_ξ are those linear combinations of vectors in U_ξ that remain to lie in U_ξ. Note that if $\sum \nu_i = 0$ and $p_i \in U_\xi$, then $\sum \nu_i p_i \in \operatorname{Ann}\xi$ is well defined vector of V. A choice of an origin $o \in U_\xi$ provides a decomposition

$$W^* = \operatorname{Ann}(o) \oplus \Bbbk \cdot \xi , \tag{5-4}$$

where $\alpha = (\alpha - \langle \alpha , o \rangle \xi) + \langle \alpha , o \rangle \xi$ for any $\alpha \in W^*$. Note that $V^* \simeq W^*/\Bbbk \cdot \xi$ is naturally isomorphic to $\operatorname{Ann}(o)$.

Exercise 5.3. Verify that the mapping $V^* = W^*/\mathbb{R} \cdot \xi \to \operatorname{Ann}(o) \subset W^*$ sending

$$\psi \pmod{\xi} \mapsto \psi - \langle \psi , o \rangle \xi$$

is well defined isomorphism of vector spaces.

5.1.3 Affine functionals. The restrictions of linear forms $\alpha \in W^*$ onto U_ξ are called *affine functionals* on U_ξ. The distinguished functional ξ is restricted to a constant function identically equal to 1. By this reason we will often write 1 instead of ξ. As soon as some origin $o \in U_\xi$ is chosen, each affine functional $\alpha|_{U_\xi}$ can be written as

$$\langle \alpha , p \rangle = \left\langle D_\alpha , \overrightarrow{op} \right\rangle + \langle \alpha , o \rangle , \quad \text{where} \quad D_\alpha \stackrel{\text{def}}{=} \alpha \pmod{\xi} \in V^* . \tag{5-5}$$

This agrees with the decomposition (5-4). Although the constant term $\langle \alpha , o \rangle$ and vector $\overrightarrow{op}$ in (5-5) do depend on a choice of the origin the linear form $D_\alpha = \alpha \pmod{\xi} \in V^*$ does not. It is called *the differential* of α and can be considered as an element $\alpha - \langle \alpha , o \rangle \in \operatorname{Ann}(o) \subset W^*$ by the identification from exrs. 5.3. Term «differential» agrees with one used in the calculus: D_α is the linear mapping $V \to \Bbbk$ such that $\forall\, x_1, x_2 \in \mathbb{A}(V) \quad \alpha(x_2) = \alpha(x_1) + D_\alpha(\overrightarrow{x_1 x_2})$, that is even more strong than in the definition of the differential of a function[1].

5.1.4 Real half-spaces. Starting from this point we assume that the ground field $\Bbbk = \mathbb{R}$.

Restricting an affine functional $\alpha : \mathbb{A}^n \to \mathbb{R}$ onto a segment $[a, b] \subset \mathbb{A}^n$, we get a linear (non-homogeneous) function $\alpha(x) = kx + c$ on the segment. There is the following alternative:

- α vanishes identically on $[a, b]$
- α is strictly positive or strictly negative everywhere on $[a, b]$, in particular, nowhere vanishes
- α vanishes at precisely one point $x_0 \in [a, b]$ and this case leads to the further alternative:
    - x_0 coincides with either a or b and α is strictly positive or strictly negative everywhere on half-open interval $[a, b] \smallsetminus x_0$

[1] recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *differentiable* at a point $x_1 \in \mathbb{A}(\mathbb{R}^n)$ if there exists a linear map $D_{f,x_1} : \mathbb{R}^n \to \mathbb{R}$, which is called *a differential* of f at x_1 and may depend on x_1 , such that for any x_2 in some neighbourhood of x_1 one has $f(x_2) = f(x_1) + D_{f,x_1}(\overrightarrow{x_1 x_2}) + o\big(|\overrightarrow{x_1 x_2}|\big)$

- $a < x_0 < b$ and α has opposite constant signs on half-open intervals $[a, x_0)$ and $(x_0, b]$, in particular, $\alpha(a) \cdot \alpha(b) < 0$.

This alternative implies that each non-constant affine functional $\alpha : \mathbb{A}^n \to \mathbb{R}$ breaks $\mathbb{A}^n$ to 3 disjunct pieces: affine hyperplane

$$H_\alpha \overset{\text{def}}{=} V(\alpha) = \{p \in \mathbb{A}^n \mid \langle\, \alpha\,,\, p\,\rangle = 0\}\,,$$

and two *open half-spaces*

$$\mathring{H}_\alpha^+ = \{p \in \mathbb{R}^n \mid \alpha(p) > 0\} \quad \text{and} \quad \mathring{H}_\alpha^- = \{p \in \mathbb{R}^n \mid \alpha(p) < 0\} \tag{5-6}$$

whose common boundary $\partial \mathring{H}_\alpha^+ = \partial \mathring{H}_\alpha^- = H_\alpha$. Each segment joining two points of the different open half-spaces intersects the boundary hyperplane in a unique point, which lies in the interior of the segment. The closures of the open half-spaces (5-6)

$$H_\alpha^+ = \{p \in \mathbb{R}^n \mid \alpha(p) \geqslant 0\} \quad \text{and} \quad H_\alpha^- = \{p \in \mathbb{R}^n \mid \alpha(p) \leqslant 0\}\,. \tag{5-7}$$

are called *closed half-spaces.*

5.2 Convex figures. A barycentric combination of points in an affine space $\mathbb{R}^n$

$$x_1p_1 + x_2p_2 + \cdots + x_mp_m\,, \quad \text{where} \quad \sum x_i = 1\,,$$

is called *convex* if all $x_i \geqslant 0$. A figure $\Phi \subset \mathbb{R}^n$ is called *convex* as soon it contains all convex combinations of its points. Clearly, an intersection of convex figures is a convex figure. Given an arbitrary figure Ψ, the intersection of all convex figures that contain Ψ is called *a convex hull* of Ψ and is denoted by $\operatorname{conv}(\Psi)$.

Exercise 5.4. Check that open half-spaces (5-6) and closed half-spaces (5-7) are convex.

Lemma 5.1
A convex combination $\sum\limits_i y_iq_i$ of convex combinations $q_i = \sum\limits_j x_{ij}p_j$ of some points p_i is again a convex combination of p_i.

Proof. $\sum\limits_{i,j} y_ix_{ij}p_j = \sum\limits_j \left(\sum_i y_ix_{ij}\right)p_j = \sum\limits_j \lambda_jp_j$, where $\lambda_j = \sum_i y_ix_{ij} \geqslant 0$ satisfy $\sum\limits_j \lambda_j = \sum\limits_{i,j} y_ix_{ij} = \sum\limits_i y_i \cdot \left(\sum_j x_{ij}\right) = \sum_i y_i = 1$. □

Corollary 5.1
A set of all convex combinations of points of an arbitrary figure Ψ is convex. In particular, $\operatorname{conv}(\Psi)$ consists of all convex combinations of points of Ψ.

Example 5.1 (simplexes)
Assume that $k+1$ points $p_0, p_1, \dots, p_k$ do not lie in a $(k-1)$-dimensional affine subspace. Then their convex hull

$$[p_0, p_1, \dots, p_k] \overset{\text{def}}{=} \left\{ \sum_{i=0}^n x_ip_i \;\middle|\; \sum_{i=0}^n x_i = 1\,,\; x_i \geqslant 0 \right\}. \tag{5-8}$$

is called a *k-dimensional simplex* with vertexes $p_0, p_1, \dots, p_k$. Thus, 1-, 2-, and 3-dimensional simplexes are the segments $[p_0, p_1] = \{\lambda_0 p_0 + \lambda_1 p_1 \mid \lambda_0, \lambda_1 \geqslant 0 \ \&\ \lambda_0 + \lambda_1 = 1\}$, the triangles $[p_0, p_1, p_2] = \triangle\, p_0 p_1 p_2$, and the tetrahedrons respectively. In affine coordinates $(x_1, x_2, \dots, x_n)$ with the origin at p_0 and basic vectors $e_i = \overrightarrow{p_0 p_i}$, $i = 1, 2, \dots, n$, the simplex (5-8) is given by a system of $(n+1)$ linear (non-homogeneous) inequalities $x_1 \geqslant 0$, $x_2 \geqslant 0$, ..., $x_n \geqslant 0$, $x_1 + x_2 + \cdots + x_n \leqslant 1$. Since the point $\left(\frac{1}{2n}, \frac{1}{2n}, \dots, \frac{1}{2n}\right)$ satisfies the strict versions of these inequalities, some cubic ε-neighbourhood of this point belongs to the simplex. Thus, the convex hull of $(n+1)$ points of $\mathbb{R}^n$ has non-empty interior as soon the points do not lie in a hyperplane.

Exercise 5.5. Show that the boundary of simplex $[p_0, p_1, \dots, p_n]$ is the union of all simplexes $[p_{\nu_1}, p_{\nu_2}, \dots, p_{\nu_m}]$, where $m < n$ and $\nu_i \in \{0, 1, \dots, n\}$.

Lemma 5.2
A figure $\Phi \subset \mathbb{R}^n$ is convex iff $\forall\, a, b \in \Phi$ the segment $[a, b] \in \Phi$.

Proof. Implication $\Rightarrow$ is trivial. To establish $\Leftarrow$, it is enough to check that any convex combination $\lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_m p_m$ can be computed step by step via taking a convex combination of appropriate two points at each step. This follows from the identity

$$\lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_m p_m = \alpha p_0 + \beta \left(\mu_2 p_2 + \mu_3 p_3 + \cdots + \mu_m p_m\right),$$

where $\alpha = \lambda_1$, $\beta = \lambda_2 + \lambda_3 + \cdots + \lambda_n$, $\mu_i = \lambda_i / \beta$. It shows that $\sum \lambda_i p_i$ is a convex combination of two points: p_0 and $q = \mu_2 p_2 + \mu_3 p_3 + \cdots + \mu_m p_m$. By induction in m, the latter point can be reached by taking consequent convex combinations of pairs of points. □

Proposition 5.1
If $\Psi \subset \mathbb{R}^n$ is convex, then its closure $\overline{\Psi}$ and its interior Ψ° are convex as well.

Proof. If $a, b \in \overline{\Psi}$, then $a = \lim_{k\to\infty} a_k$ and $b = \lim_{k\to\infty} b_k$ for some sequences $\{a_k\}, \{b_k\} \subset \Psi$. For any fixed non-negative λ, μ with $\lambda + \mu = 1$ we have

$$\lim_{k\to\infty} (\lambda a_k + \mu b_k) = \lambda \lim_{k\to\infty} a_k + \mu \lim_{k\to\infty} b_k = \lambda a + \mu b\,.$$

Hence, all points of the segment $[a, b]$ are the limit points of some sequences in Φ. Thus, $[a, b] \subset \overline{\Psi}$.

If $a, b \in \Psi^\circ$, then their cubic ε-neighbourhoods $B_\varepsilon(a)$, $B_\varepsilon(b)$ are contained in Φ for some $\varepsilon > 0$. Thus, cubic ε-neighbourhoods of all points of the segment $[ab]$ are contained in Φ as well, see fig. 5⋄1. □

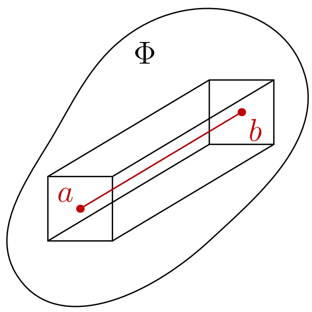


Fig. 5⋄1. Convexity of the interior.

Exercise 5.6. Show that each convex closed figure Φ coincides with the closure of its interior: $\Phi = \overline{\Phi^\circ}$ (in the topology of the smallest affine space containing Φ).

5.2.1 Supporting half-spaces. A hyperplane H_α is called *supporting* a figure Φ if $\Phi \subset H_\alpha^+$ and $H_\alpha \cap \partial\Phi \neq \varnothing$. In this case H_α^+ is also called *supporting half-space* of Φ.

Lemma 5.3
In affine space $\mathbb{A}^n$ of dimension $n \geqslant 2$ for any open convex set $U \subset \mathbb{A}^n$ and any point $p \notin U$ there exists a line ℓ that passes through p and does not intersect U.

Proof. Write $C \subset \mathbb{A}^n$ for a union of all open rays $]p, u) \overset{\text{def}}{=} \{p + \lambda \cdot \overrightarrow{pu} \mid \lambda > 0\}$ passing through all $u \in U$. It is evident from fig. 5◇2 and fig. 5◇3 that C is open and convex, and $p \in \partial C$. Convexity of C implies that each line passing through p either does not intersect U at all or intersects U in such a way that all points from one side of p lie in the interior of C and all points from the other side of p lie in the exterior[1] C, see fig. 5◇2. In particular, there exists a point q exterior for C. Since $n > 1$, there is a line passing though q, intersecting C, and different from (qp). This line must contain a point $r \in \partial C$. Then, the line (pr) passes through p and does not intersect U and even C, because it contains a boundary point of C different from p. □

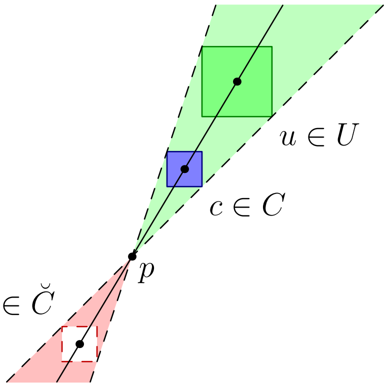


Рис. 5◇2. Openness of C and exterior points.

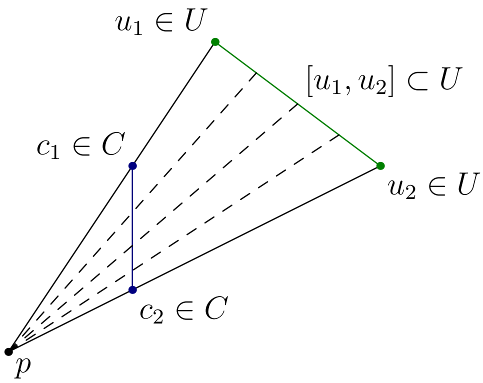


Рис. 5◇3. Convexity of C.

Lemma 5.4
Let an affine plane Π (possibly zero dimensional) do not intersect an open convex set U. Then there exists a hyperplane (of codimension 1) that contains Π and does not intersect U.

Proof. Chose the origin of $\mathbb{A}^n$ inside Π and identify $\mathbb{A}^n$ with V. Under this identification Π goes to some vector subspace $P \subset V$ ($P = 0$ is allowed as well). In the set of all vector subspaces $H \subset V$ that contain P and do not intersect U chose some maximal w.r.t. inclusions subspace $H \subset V$. Pick up some vector subspace $H' \subset V$ such that $V = H \oplus H'$. We claim that $\dim H' = 1$. Indeed, let $\pi : V \twoheadrightarrow H'$ be the projection along H. Since the projection takes segments and cubes to segments and cubes, $\pi(U)$ is an open convex set and $0 \notin \pi(U)$, because $H \cap U = \varnothing$. If $\dim H' > 1$, then by lemma 5.3 there exists 1-dimensional subspace $L \subset H'$ such that $L \cap \pi(U) = \varnothing$. Hence, $H \oplus L$ does not intersect U, contains P, and is strictly bigger than H contrary to the choice of H. □

Theorem 5.1
For any convex figure Φ and any point $p \in \partial\Phi$ there exists a supporting hyperplane of Φ passing through p.

[1]i.e. in the interior of $\mathbb{A}^n \smallsetminus U$

Proof. If Φ lies in some hyperplane, then this hyperplane is supporting for Φ. If Φ contains $(n+1)$ points that do not lie in a common hyperplane, then it follows from example 5.1 that the interior Φ° of Φ is not empty. Draw a hyperplane H_α that passes through p and does not intersect Φ°. Since α vanish nowhere in Φ°, the interior of Φ is contained inside one of two open half-spaces (5-6). Changing a sign of α, if required, we can assume that $\Phi^\circ \subset H_\alpha^+$. Hence, by exrs. 5.6, $\Phi \subset \overline{\Phi^\circ} \subset \overline{H_\alpha^{+\circ}} = H_\alpha^+$. □

Theorem 5.2
Any closed convex figure Z coincides with the intersection of its supporting half-spaces.

Proof. Induction in the dimension of the smallest affine subspace that contains Z allows us to assume that Z is not contained in a hyperplane, that is, has non-empty interior. We claim that for any $q \notin Z$ there exists a supporting hyperplane H_α, of Z, such that $q \in H_\alpha^-$. To construct it, let us join q with some interior point $p \in Z^\circ$ by the segment $[q,p]$. Then there should be a boundary point $r \in [q,p] \cap \partial Z$ laying in the interior of $[q,p]$. But the relations $\langle\, \alpha\,,\, p\,\rangle > 0$ and $\langle\, \alpha\,,\, r\,\rangle = 0$ force $\langle\, \alpha\,,\, q\,\rangle < 0$. □

5.2.2 Faces and extremal points. Let Φ be a closed convex figure and H_α be a supporting hyperplane of Φ. In this situation $H_\alpha \cap \Phi$ is a non-empty closed convex figure. It is called *a face* of Φ. Given a face $\Gamma \subset \Psi$, we write $\mathbb{A}^\Gamma$ for the smallest affine subspace containing Γ and call it *affine hull* of the face Γ. On default, we consider each face Γ in the standard topology of $\mathbb{A}^\Gamma$. Interior, exterior, and boundary points of Γ are those w.r.t. this default topology, and we put $\dim \Gamma \overset{\text{def}}{=} \dim \mathbb{A}^\Gamma$. Zero dimensional face, that is, a face exhausted by one point, is called *a vertex*.

Note that faces of an arbitrary closed convex figure may behave quite counter-intuitively. For example, a closed ball has continual family of distinct faces and all these faces are the vertexes. A smooth mating of half-circle with rectangle shown in fig. 5◇4 has two 1-dimensional faces whose vertexes are not the vertexes of the initial figure. Thus a face of a face is not necessary a face of the initial figure.

Fig. 5◇4.

Points that can be reached as the last, zero dimensional, elements of chains: a figure Φ, a face Φ_1 of Φ, a face Φ_2 of the face Φ_1, a face Φ_3 of the face Φ_2, … are called *extremal points* of a closed convex figure Φ. This definition implies that extremal points of any face of Φ are the extremal points of Φ itself. In particular, each vertex is an extremal point.

Exercise 5.7. Give an example of closed convex figure Φ whose extremal points are not exhausted by the vertexes of Φ.

Lemma 5.5
Let Φ be a closed convex figure and $p \in \Phi$. Then p is extremal point of Φ iff there are no segments $[a,b] \subset \Phi$ containing p in the interior.

Proof. If $p \in \Phi$ is interior point of Φ or some face $\Gamma \subset \Phi$ such that $\dim \Gamma \geqslant 1$, then there exists a segment $[a,b] \subset \Gamma$ containing p in the interior. Thus, if there are no segments $[a,b] \subset \Phi$ containing p in the interior, then p is neither in the interior of Φ, nor in the interior of a face of Φ, nor in the interior of a face of a face of Φ, etc as far the dimension of face remains greater than zero. Hence, p is extremal.

Vice versa, if p lies in the interior of a segment $[a,b] \subset \Phi$, then p belongs to some face $\Gamma \subset \Phi$ only if $[a,b] \subset \Gamma$. Indeed, let $\Gamma = \Phi \cap H_\eta$ for some supporting functional η. If $p \in \Gamma$, then

$\langle \eta , p \rangle = 0$ and $\langle \eta , a \rangle \geqslant 0$ and $\langle \eta , b \rangle \geqslant 0$. This forces $\langle \eta , a \rangle = \langle \eta , b \rangle = 0$, i.e. $[a,b] \subset \Gamma$. We conclude that p can not be a vertex of Φ as well as a vertex of a face of Φ, as well as a vertex of a face of a face of Φ etc. Thus, p is not extremal. □

Proposition 5.2
Each compact closed convex figure Φ coincides with a convex hull of its extremal points.

Proof. Induction in $\dim \Phi$. Each interior point $p \in \Phi^\circ$ is a convex combination of the endpoints of a segment $\ell \cap \Phi$, where ℓ is an arbitrary line passing through p. These endpoints lie on some faces. By induction, they are convex combinations of the extremal points of these faces. But extremal points of faces are the extremal points of Φ as well. □

5.3 Convex polyhedrons. An intersection of a finite number of half-spaces

$$M_A = \bigcap_{\alpha \in A} H_\alpha^+ , \quad \text{where } A = \{\alpha_1, \alpha_2, \dots , \alpha_m\} \subset W^*,$$

is called *a convex polyhedron.* In this definition it is convenient to allow «non-proper half-spaces»

$$H_\xi = H_1 = U_\xi = \mathbb{A}^n \quad \text{and} \quad H_{-\xi} = H_{-1} = \varnothing$$

as well. Each convex polyhedron is convex and closed. It may be empty or may coincide with the whole space $\mathbb{A}^n$. An intersection of any finite collection of convex polyhedrons is a convex polyhedron too.

Exercise 5.8. Verify that all affine subspaces (including the empty set, the points, and the whole space) are convex polyhedrons.

In particular, an intersection of a convex polyhedron with an affine subspace is a convex polyhedron.

5.3.1 Faces of polyhedrons. It is convenient to consider each convex polyhedron M as its own face[1]. We call this special face $\Gamma = M$ *the non-proper face* in a contrast with *proper faces* $\Gamma \subsetneq M$ whose dimension is strictly less than $\dim M$. Each proper[2] polyhedron has proper faces. Faces $\Gamma \subset M$ of dimension $\dim \Gamma = \dim M - 1$ are called *hyperfaces.*

Proposition 5.3 (geometric description of faces)
Let a polyhedron $M = \bigcap_{\alpha \in A} H_\alpha^+$ be defined by a finite set $A \subset W^*$ of affine functionals. For each non-empty subset $I \subseteq A$ we put $H_I \overset{\text{def}}{=} \bigcap_{\alpha \in I} H_\alpha$ and $\Gamma_I \overset{\text{def}}{=} M \cap H_I$. For each face $\Gamma \subset M$ we put $A_\Gamma \overset{\text{def}}{=} \{\alpha \in A \mid \Gamma \subseteq H_\alpha\}$ and $H_\Gamma \overset{\text{def}}{=} H_{A_\Gamma}$. Then

1) $\forall\, I \subseteq A \;\; \Gamma_I \subset M$ is either empty or a face of M

2) for each proper face $\Gamma \subset M \;\; A_\Gamma \neq \varnothing$ and $\Gamma = \Gamma_{A_\Gamma}$

3) a point p of a face $\Gamma \subseteq M$ lies in the interior[3] of Γ iff $\forall\, \alpha \notin A_\Gamma \;\; \alpha(p) > 0$

[1] one can think of this face as being cut out of M by the «non-proper half-space» H_ξ
[2] that is non-empty and distinct from the whole space
[3] in the topology of the affine hull $\mathbb{A}^\Gamma$

4) for any face $\Gamma \subseteq M$ $H_\Gamma = \mathbb{A}^\Gamma$ is the affine hull of Γ.

Proof. (1) is obvious: if $\Gamma_I = M \cap H_I \neq \varnothing$, then affine functional $\alpha_I = \sum\limits_{\alpha \in I} \alpha$ is supporting for M and $\Gamma_I = M \cap H_{\alpha_I}$.

Now consider an arbitrary face $\Gamma \subseteq M$. For each $\psi \in A \smallsetminus A_\Gamma$ there exists a point $q_\psi \in \Gamma$ such that $\psi(q_\psi) > 0$. Write q_Γ for the barycentre of all these points. Then all $\psi \in A \smallsetminus A_\Gamma$ are strictly positive at q_Γ. If $A_\Gamma = \varnothing$, then all $\alpha \in A$ are strictly positive at q_Γ. Hence, they are strictly positive in some neighbourhood of q_Γ in the topology of the whole space, i.e. q_Γ lies in the interior of M and can not appear in a proper face. Thus for each proper face Γ we have the inclusion $\Gamma \subset H_\Gamma \cap M = \Gamma_{A_\Gamma}$ and the inequalities $\Gamma_{A_\Gamma} \neq \varnothing$, $A_\Gamma \neq \varnothing$, and $H_\Gamma \neq \varnothing$.

Let us prove that for each «special» face Γ_{A_Γ}, where $G \subset M$ is some face, the statement (3) holds (including non-proper case $\Gamma_{A_M} = M$).

Firstly, assume that $p \in \Gamma_{A_\Gamma} = H_\Gamma \cap M$ satisfies $\psi(p) > 0$ for all $\psi \in A \smallsetminus A_\Gamma$. Then these strict inequalities hold in some neighbourhood of p in the topology of H_Γ. Hence, this neighbourhood is contained in $\Gamma_{A_\Gamma} = H_\Gamma \cap M$. This implies that p lies in the interior of face Γ_{A_Γ} as well as that H_Γ is the smallest affine space containing Γ_{A_Γ}. The latter means that (4) automatically follows from (2) and (3).

Secondary, assume that $\psi \in A$ satisfies $\psi(p) = 0$ and $\psi(q) > 0$ for some points p, q laying in some face $\Gamma' \subseteq M$. Then p can not lie in the interior of Γ', because otherwise we could extend a segment $[q, p]$ out of p and get a point $r \in \Gamma'$ such that $\psi(r) < 0$.

Now we are ready to prove (2) and (3) for any face $\Gamma = M \cap H_\eta$, $H_\eta^+ \supset M$. The point $q_\Gamma \in \Gamma \subset \Gamma_{A_\Gamma}$ described above is an interior point of the face Γ_{A_Γ}. For each point $p \in \Gamma_{A_\Gamma}$, $p \neq q_\Gamma$ we can extend the segment $[p, q_\Gamma]$ little bit out of q_Γ to get a segment $[p, r]$ such that $q_\Gamma \in [p, r] \subset \Gamma_{A_\Gamma}$. Then the relations $\eta(q_\Gamma) = 0$, $\eta(p) \geqslant 0$, and $\eta(r) \geqslant 0$ imply that $\eta(r) = \eta(p) = 0$. Thus, $p \in \Gamma$ and $\Gamma_{A_\Gamma} \subset \Gamma$. This proves (2) as well as (3), and automatically (4). ☐

Corollary 5.2
Any convex polyhedron has a finite collection of faces, all the faces are convex polyhedrons, and each face of any face is a face of the initial polyhedron. ☐

Corollary 5.3
The extremal points of a convex polyhedron are exhausted by its vertexes. ☐

Corollary 5.4
Each compact convex polyhedron has a finite collection of vertexes and coincides with their convex hull. ☐

Proposition 5.4
Following properties of a convex polyhedron $M \subset \mathbb{A}(V)$ are mutually equivalent:

1) M has no vertexes

2) M contains an affine subspace of positive dimension

3) $V = U \oplus W$, where $0 < \dim U < \dim V$, and $M = \mathbb{A}(U) \times M' \subset \mathbb{A}(U) \times \mathbb{A}(W)$, where $M' \subset \mathbb{A}(W)$ is a convex polyhedron of dimension $\dim M' = \dim M - \dim U < \dim M$.

Proof. (1) ⇒ (2). If M does not coincide with its affine span L, then M has a supporting hyperplane inside L and a proper face $\Phi \subsetneq M$ of dimension less than $\dim M$. Replacing M by Φ and repeating the arguments we come to a face of M that coincides with its affine span. If this span is a point, then the face is a vertex of M. Thus, (1) implies (2).

(2) ⇒ (3). Let $M = \bigcap_{\alpha\in A} H_\alpha^+$ contain an affine subspace $p + U$, where $U \subset V$ is a non-zero vector subspace. We claim that the differentials $\psi_\alpha = D_\alpha$ of all linear functionals $\alpha \in A$ lie in $\operatorname{Ann} U$. Inded, if there are some $u \in U$ and $\alpha \in A$ such that $\psi_\alpha(u) < 0$, then for $t \gg 0$ inequality $\alpha(p + tu) = t\psi_\alpha(u) + \psi(p) + c_\alpha < 0$ holds. It means that $p + tu \notin M$.

Pick up some $W \subset V$ such that $V = U \oplus W$ and $\mathbb{A}(V) = \mathbb{A}(U) \times \mathbb{A}(W)$. Since $\forall\, \alpha \in A$ and $\forall\,(u, w) \in \mathbb{A}(U) \times \mathbb{A}(W)$ the value $\alpha(u, w) = \alpha(o, w)$ does not depend on $u \in \mathbb{A}(U)$, we have $M = \mathbb{A}(U) \times M'$, where $M' \subset \mathbb{A}(W)$ is defined by the restriction of inequalities $\alpha(v) \geqslant 0$ onto affine subspace $\{o\} \times \mathbb{A}(W) \subset \mathbb{A}(V)$.

(3) ⇒ (1). If $M = \mathbb{A}^k \times M'$, where $k \geqslant 1$, then each point of M lies in affine subspace $\mathbb{A}^k \subset M$. This means that M has no extremal points. □

5.4 Convex polyhedral cones. Given a point $o \in \mathbb{A}(V)$ and a finite collection of vectors $v_1, v_2, \dots, v_m \in V$, a set $\sigma_{v_1,v_2,\dots,v_m} = \{o + \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_m v_m \in V \mid \forall\, i\ \lambda_i \geqslant 0\}$ is called *a convex polyhedral cone* spanned by $v_1, v_2, \dots, v_m \in V$ at o. On default we shorten this title just to *a cone* on generators v_i and identify o with zero $0 \in V$. We call o *the vertex of the cone*, although it may be even not a face of the cone at all.

Exercise 5.9. Verify that any cone σ with vertex o is convex and closed and for each $p \in \sigma$ the whole of closed ray $[o, p) = \{o + \lambda\overrightarrow{op} \mid \lambda \geqslant 0\}$ belongs to σ.

Lemma 5.6
Each supporting hyperplane of any convex polyhedral cone passes through its vertex.

Proof. Write o for the vertex of cone σ ant let a supporting hyperplane H_η of σ pass through a point $p \in \partial\sigma$, $p \neq o$. If $\eta(o) > 0$, then the restriction of η onto the ray $[o, p) \subset \sigma$ has to change a sign at p. This is impossible by exrs. 5.9. □

Corollary 5.5 (the Farkas lemma)
Assume that the vertex of a cone $\sigma_{v_1,v_2,\dots,v_m} \subset \mathbb{A}(V)$ is taken at the origin $0 \in V$. Then for any $u \notin \sigma_{v_1,v_2,\dots,v_m}$ there exists $\varphi \in V^*$ such that $\varphi(u) < 0$ and $\forall\, i\ \ \varphi(v_i) \geqslant 0$.

Proof. By theorem 5.2 and lemma 5.6 cone $\sigma_{v_1,v_2,\dots,v_m}$ coincides with the intersection of its supporting half-spaces $H_\varphi^+ = \{v \in V \mid \varphi(v) \geqslant 0\}$, where $\varphi \in V^*$. Thus, for any $u \notin \sigma_{v_1,v_2,\dots,v_m}$ there exists $\varphi \in V^*$ such that $\sigma_{v_1,v_2,\dots,v_m} \subset H_\varphi^+$ but $u \notin H_\varphi^+$. □

Theorem 5.3 (Farkas – Minkowski – Weyl)
Each convex polyhedral cone σ is an intersection of a finite collection of half-spaces passing through the vertex of σ and vice versa.

Proof. Let $\Phi \subset V$ be an intersection of a finite collection of half-spaces passing through the origin $0 \in V$. Then Φ is a convex polyhedron and for each non zero $v \in \Phi$ the whole of closed ray $[0, v) = \{\lambda v \mid \lambda \geqslant 0\}$ belongs to Φ. Write I^n for the standard unit cube centred at the origin. Then $\Phi \cap I^n$ is compact convex polyhedron. By cor. 5.4 it coincides with a convex hull of a finite collection of vectors $v_1, v_2, \dots, v_m \in \Phi \cap I^n$. Since some positive multiplicity λv of any $v \in \Phi$ lies in $\Phi \cap I^n$, each $v \in \Phi$ is a positive linear combination of the vectors v_i. Hence, $\Phi = \sigma_{v_1, v_2, \dots, v_m}$.

Vice versa, each cone $\sigma_{v_1, v_2, \dots, v_m} \subset V$ with vertex at the origin either coincides with V or is an intersection of supporting half-spaces $H_\psi^+ = \{v \in V \mid \psi(v) \geqslant 0\}$, where $\psi \in V^*$ runs through the set $\Psi_\sigma = \left\{\psi \in V^* \mid \left\langle \psi, v_i \right\rangle \geqslant 0, \ i = 1, 2, \dots, m\right\}$, which is an intersection of half-spaces $H_{v_i}^+ \subset V^*$ passing through the origin of V^*. We already have proven that $\Psi_\sigma = \sigma_{\psi_1, \psi_2, \dots, \psi_N} \subset V^*$ for appropriate $\psi_1, \psi_2, \dots, \psi_N \in V^*$. Thus, all inequalities $\psi(v) \geqslant 0$, $\psi \in \Psi_\sigma$, follow from the finite collection of inequalities $\psi_\nu(v) \geqslant 0$, $\nu = 1, 2, \dots, N$, i.e. $\sigma_{v_1, v_2, \dots, v_m} = \bigcap\limits_{\nu=1}^{N} H_{\psi_\nu}^+$. □

Corollary 5.6 (Minkowski – Weyl)
A convex hull of any finite collection of points is a compact convex polyhedron and vice versa.

Proof. A convex hull $M = \operatorname{conv}(p_1, p_2, \dots, p_m)$ of an arbitrary finite collection of points $p_1, p_2, \dots, p_m$ sits inside any cube containing all the points p_i. On the other side, it coincides with the intersection of convex polyhedral cone $\sigma_{p_1, p_2, \dots, p_m} \subset \mathbb{A}(W)$ with affine hyperplane U_ξ. Thus, M is a compact polyhedron. The inverse statement was already proven in cor. 5.4. □

5.4.1 Duality. Consider a finite set of non-zero vectors $R \subset V$ and write $\sigma_R \subset V$ for a cone with vertex at the origin spanned by these vectors. The set of functionals supporting σ_R is called *a dual cone* and is denoted by $\sigma_R^\vee \subset V^*$. The dual cone has tautological presentation via intersection of vector half-spaces

$$\sigma_R^\vee = \{\psi \in V^* \mid \forall\, v \in \sigma_R \ \langle \psi, v \rangle \geqslant 0\} = \bigcap_{v \in R} H_v^+ .$$

By theorem 5.3 this intersection is a cone $\sigma_R^\vee = \sigma_{R^\vee}$ spanned by some finite collection of covectors $R^\vee \subset V^*$. By cor. 5.5 the initial cone σ_R is dual to $\sigma_R^\vee$:

$$\sigma_R = \{v \in V \mid \forall\, \psi \in \sigma_R^\vee \ \langle \psi, v \rangle \geqslant 0\} = \bigcap_{\psi \in R^\vee} H_\psi^+ = \sigma_{R^\vee}^\vee$$

Thus, for any convex polyhedral cone σ we have coincidence $\sigma^{\vee\vee} = \sigma$.

Following prop. 5.3, we associate with each face $\Gamma \subset \sigma_R$ a subset $R_\Gamma^\vee \subset R^\vee$ that consists of all generators of the dual cone that annihilate Γ: $R_\Gamma^\vee \stackrel{\text{def}}{=} \{\psi \in R^\vee \mid \Gamma \subset H_\psi\} = R^\vee \cap \operatorname{Ann} \Gamma$. We also put $H_\Gamma \stackrel{\text{def}}{=} \bigcap\limits_{\psi \in R_\Gamma^\vee} H_\psi = \operatorname{Ann} R_\Gamma^\vee$.

Corollary 5.7
Let $\sigma_R \subset V$ be a convex polyhedral cone with vertex at zero. Then each face $\Gamma = \sigma_R \cap H_\Gamma$ is a cone spanned by those vectors of R that lie in Γ, i.e. $\Gamma = \sigma_{R_\Gamma}$, where

$$R_\Gamma \stackrel{\text{def}}{=} R \cap \Gamma = R \cap \left(\sigma_R \cap H_\Gamma\right) = R \cap H_\Gamma .$$

Moreover, the collection of vectors R_Γ linearly spans the subspace H_Γ spanned by Γ.

Proof. The subspace H_Γ is linearly spanned by Γ by prop. 5.3. Moreover, $\Gamma = \sigma_R \cap H_\Gamma \supset R_\Gamma$. Thus, we have only to show that each vector of Γ can be written as *non-negative* linear combination of vectors from R_Γ. For any generator $v \in R \smallsetminus R_\Gamma$ there is a covector $\psi \in R_\Gamma^\vee$ such that $\langle \psi , v \rangle > 0$. Hence this generator can not appear with positive coefficient in the expansions of $w \in \Gamma$ through the generators of the cone, because evaluation at $w \in \Gamma$ annihilates all $\psi \in R_\Gamma^\vee$. □

Exercise 5.10. Give an example of a cone σ_R and non-empty subset $I \subset R$ such that σ_I is not a face (even non-proper) of σ_R.

Corollary 5.8
For any $k = 0, 1, \dots , \dim \sigma$ there is a bijection between k-dimensional faces of a cone $\sigma_R \subset V$ and $(n-k)$-dimensional faces of the dual cone $\sigma_R^\vee = \sigma_{R^\vee} \subset V^*$, where $n = \dim V$ and «$\dim \sigma$-dimensional face of σ» means the cone σ itself. This bijection reverses the inclusions and is given by the rule $\Gamma = \sigma_R \cap \operatorname{Ann} \Gamma^\vee = \sigma_R \cap H_\Gamma = \sigma_{R_\Gamma} \leftrightarrows \sigma_{R_\Gamma^\vee} = \sigma_{R^\vee} \cap \operatorname{Ann} H_\Gamma = \sigma_{R^\vee} \cap \operatorname{Ann} \Gamma = \Gamma^\vee$. In particular, 1-dimensional edges of σ are precisely the equations of hyperfaces of $\sigma^\vee$ and vice versa.

Proof. The linear span of $R_\Gamma^\vee$ coincides with the double annihilator $\operatorname{Ann}\operatorname{Ann}(R_\Gamma^\vee) = \operatorname{Ann} H_\Gamma = \operatorname{Ann} \Gamma$. By prop. 5.3 on p. 90, this subspace cuts the face

$$\Gamma^\vee = \operatorname{Ann} H_\Gamma \cap \sigma_{R^\vee} = \sigma_{R_\Gamma^\vee} ,$$

of the dual cone $\sigma_R^\vee = \sigma_{R^\vee}$. By cor. 5.8, this face is a cone spanned by $R_\Gamma^\vee = \operatorname{Ann} H_\Gamma \cap R^\vee$. □

Remark 5.1. We *do not assume* in cor. 5.8 that $\dim \sigma_R = \dim V$. For example, 1-dimensional cone $\sigma_v = \{tv \mid t \geqslant 0\}$ (in any vector space V) is a row going out of zero in direction of vector v. It has two faces: zero-dimensional face at zero and 1-dimensional face coinciding with the row itself. The dual cone $\sigma_v^\vee = H_v^+ \subset V^*$ is a half-space and also has two faces: non-proper n-dimensional face $\sigma_v^\vee \cap \operatorname{Ann} o = H_v^+ \cap V^* = H_v^+$ and $(n-1)$-dimensional face $\operatorname{Ann} v = H_v$.

Exercise 5.11. For each face $\Gamma = \sigma_{R_\Gamma}$ of an arbitrary polyhedral cone σ_R verify the identity $\sigma_{R_\Gamma} = \sigma \cap \left(-\sigma^\vee_{R_\Gamma^\vee}\right)$, where $-\sigma = \{v \mid -v \in \sigma\}$.

5.4.2 Affine and asymptotic cones of polyhedron. Assume that we are given with a finite set of covectors $A \subset W^*$ that includes the distinguished covector ξ whose restriction onto $\mathbb{A}^n = U_\xi \subset W$ identically equals 1. We write

$$M_A = \bigcap_{\alpha \in A} H_\alpha^+ \subset \mathbb{A}(V)$$

for the corresponding polyhedron in $\mathbb{A}^n = U_\xi$ (it may be empty or coincide with the whole of $\mathbb{A}^n$). Set A spans convex polyhedral cone $\sigma_A \subset W^*$. Its dual cone is called *an affine cone* over M and is denoted by

$$\sigma_M = \sigma_A^\vee \subset W .$$

The intersection of the affine cone with the infinity $\operatorname{Ann} \xi = V$ of U_ξ is called *an asymptotic cone* of M and is denoted by

$$\delta_M \overset{\text{def}}{=} \sigma_A^\vee \cap \operatorname{Ann} \xi = \{v \in V \mid \forall\, \alpha \in A \ \langle \alpha , v \rangle \geqslant 0\} \subset V . \tag{5-9}$$

Note that asymptotic cone lies in the vector space $V = \operatorname{Ann} \xi$ parallel the affine space $\mathbb{A}^n = U_\xi$ and is dual to the cone spanned by differentials $D_\alpha = \alpha \pmod{\xi} \in V^*$ of the functionals $\alpha \in A$:

$$\delta_M^\vee = \sigma_A \pmod{\xi} \subset W / \Bbbk \cdot \xi = V^* . \tag{5-10}$$

Proposition 5.5
If $\xi \in A$, then $M_A = \varnothing \iff -\xi \in \sigma_A$. If $M \neq \varnothing$, then $\sigma_M \subset W$ is the closure of the union of all rays $[0, p) = \{\lambda p \mid \lambda \geqslant 0\} \subset W$ drawn from zero to the points $p \in M$.

Proof. Of course, $-\xi \in A \Rightarrow M = \varnothing$. Let $M = \varnothing$. Then $\langle \xi , w \rangle \leqslant 0$ for all $w \in \sigma_A^\vee$, because in the contrary case $\exists\, w$ such that $\langle \xi , w \rangle = 1$ and $\langle \alpha , w \rangle \geqslant 0$ for all $\alpha \in A$, i.e. $w \in M$. Thus, $-\xi$ is supporting for $\sigma_A^\vee$, i.e. $-\xi \in \sigma_A^{\vee\vee} = \sigma_A$.

If $M \neq \varnothing$, then all rays $[0, p)$ with $p \in M$ lie in σ_M. Write $C \subset W$ for the closure of the union of all these rays. Then $C \subset \sigma_M$. On the other hand, $\langle \xi , w \rangle > 0$ for each internal point w of $\sigma_M = \sigma_A^\vee$, because otherwise $\sigma_A^\vee \subset \operatorname{Ann} \xi$ and $M = \varnothing$. Thus, the row $[0, w)$ drawn through any internal point $w \in \sigma_M^\circ$ intersects M, i.e. has a form $[0, p)$ for some $p \in M$. By exrs. 5.6, $\sigma_M = \overline{\sigma_M^\circ} \subset C$. □

Exercise 5.12. Show that $M = \mathbb{A}(V) \iff \sigma_A = \sigma_\xi$ (row of non-negative constants).

Proposition 5.6
If $M_A \neq \varnothing$, then

$$\delta_M = \{v \in V \mid \forall\, p \in M\, \forall\, \lambda > 0 \ p + \lambda v \in M\} = \{v \in V \mid \exists\, p \in M : \forall\, \lambda > 0 \ p + \lambda v \in M\}\,.$$

Proof. Let $p \in M$. Then $p + \lambda v \subset M$ iff $\forall\, \alpha \in A\ \langle \alpha , p + \lambda v \rangle = \langle \alpha , p \rangle + \lambda \left\langle D_\alpha , v \right\rangle \geqslant 0$. The latter inequality holds for all $\lambda \gg 0$ iff $\left\langle D_\alpha , v \right\rangle \geqslant 0$. □

Exercise 5.13. Let $M = \mathbb{A}^m \times M'$. Show that $\delta_M = \mathbb{A}^m \times \delta_{M'}$.

Proposition 5.7
Let $\xi \in A$ and $M_A \neq \varnothing$. Then $\sigma_A = \{\alpha \in W^* \mid \forall\, p \in M_A \ \langle \alpha , p \rangle \geqslant 0\}$.

Proof. Of course, all covectors of σ_A are non-negative on M_A. Let $\eta \in W^* \smallsetminus \sigma_A$. By the Farkas lemma[1] there exists $w \in \sigma_M = \sigma_A^\vee$ such that $\langle \eta , w \rangle < 0$. Since $\xi \in \sigma_A$, $\langle \xi , w \rangle$ is not negative. If $\langle \xi , w \rangle > 0$, then $\lambda w \in U_\xi$ for some $\lambda > 0$ and hence $\lambda w \in M_A$, because $\langle \alpha , \lambda w \rangle \geqslant 0$ for all $\alpha \in A$. Thus, η takes negative value at $\lambda w \in M$. If $\langle \xi , w \rangle = 0$, then $w \in \delta_M = V \cap \sigma_A^\vee$, that is, for each $p \in M$ the ray $\{p + \lambda w \mid \lambda \geqslant 0\}$ is contained in M_A. Since $\langle \eta , p + \lambda w \rangle = \langle \eta , p \rangle + \lambda \left\langle D_\eta , w \right\rangle < 0$ for $\lambda \gg 0$, functional η takes negative values in M_A. □

Exercise 5.14. Show that M_A is compact (maybe empty) iff ξ is an interior point of σ_A.

Corollary 5.9
If $\xi \in A \subset W^*$, then for any $k = 1, 2, \dots , n = \dim V$ prescription $\tau \mapsto M_A \cap H_\tau$ establishes a bijection between k-dimensional faces of σ_A that do not contain ξ and all $(n-k)$-dimensional faces of $M_A \subset \mathbb{A}(V)$. This bijection reverses the inclusions.

Proof. By cor. 5.8, k-dimensional faces of σ_A and $(n+1-k)$-dimensional faces of $\sigma_A^\vee = \sigma_M$ stay in the bijection that reverses inclusions and takes $\sigma_A \subset \tau \mapsto \sigma_A^\vee \cap \operatorname{Ann} \tau$. On the other hand, $(n+1-k)$-dimensional face $\tau^\vee \subset \sigma_A^\vee$ either is contained in $\operatorname{Ann} \xi$ or intersects U_ξ in $(n-k)$-dimensional affine subspace. The first means that $\xi \in \tau$, the second means that $\sigma_A^\vee \cap U_\xi$ is $(n-k)$-dimensional face of $M_A = \sigma_A^\vee \cap U_\xi$. This establishes a bijection between $(n-k)$-dimensional faces of M_A and $(n+1-k)$-dimensional faces of $\sigma_A^\vee$ that are not contained in $\operatorname{Ann} \xi$, because each face $\Gamma \subset M$ is obtained in this way from the face of σ_M spanned by Γ in W. □

[1] see cor. 5.5 on p. 92

5.5 Linear programming. Given an affine functional φ and a polyhedron M, the standard problem of linear programming is either to find minimum of φ on M or to establish that φ is unbounded below on M. Similar questions about the maximum of φ are reduced to the minimization problems by the formal change of the sign: $\max \varphi = -\min(-\varphi)$.

Geometrically, it is convenient to realize V as n-dimensional vector subspace $V = \operatorname{Ann} \xi \subset W$ in $(n+1)$-dimensional vector space W on which some distinguished linear form $\xi \in W^*$ is fixed. Then $V^* = W^*/\mathbb{R}\cdot\xi$ and $\mathbb{A}^n = \mathbb{A}(V) = U_\xi = \{w \in W \mid \langle \xi , w \rangle = 1\}$.

Polyhedron $M = M_A$ is given by a finite set $A \subset W^*$ of linear forms on W considered as affine functionals on U_ξ. Without loss of generality we can and will assume that all elements of A are pairwise non-proportional and $\xi \in A$. The latter assumption means that the set of linear inequalities defining M contains the trivial inequality $1 \geqslant 0$.

It follows from prop. 5.5 that $M \neq \varnothing$ iff $-\xi \notin \sigma_A$.

Further, prop. 5.7 implies that $m \in \mathbb{R}$ is a lower boundary for φ on M iff $\varphi - m\xi \in \sigma_A$.

In particular, φ is bounded below on M iff $D_\varphi = \varphi \pmod{\xi}$ lies in the cone $\sigma_A \pmod{\xi} = \delta_M^\vee$ dual to the asymptotic cone $\delta_M \subset V$, see prop. 5.7.

Let us write A_{red} for $A \smallsetminus \xi$. The cone $\delta_M^\vee \subset V^* = W^*/\mathbb{R}\cdot\xi$ is spanned by D_α, $\alpha \in A_{\mathrm{red}}$. Any $\varphi \in W^*$ such that $D_\varphi \in \delta_M^\vee$ admits an expression

$$\varphi - m\xi = \sum_{\alpha \in A_{\mathrm{red}}} y_\alpha \cdot \alpha \tag{5-11}$$

for appropriate $m \in \mathbb{R}$ and $y_\alpha \in \mathbb{R}_{\geqslant 0}$. Hence, any such φ is bounded below by m. Since the minimum of φ on M equals the maximal lower boundary of φ on M, we conclude that

$$\min_{p \in M_A} \langle \varphi , p \rangle = \max \left(m \in \mathbb{R} \mid \varphi - m\xi \in \sigma_A \right) . \tag{5-12}$$

If $M \neq \varnothing$, i.e. $-\xi \notin A$, then the maximum in R.H.S. of (5-12) does exist, because otherwise $-\xi + \varphi/m \in \sigma_A$ for all $m \gg 0$, which implies that $-\xi \in \sigma_A$, because σ_A is closed. Summarizing, we get

Proposition 5.8
A functional $\varphi \in W^*$ is bounded below on a polyhedron $M_A = \bigcap_{\alpha \in A} H_\alpha^+$ iff $D_\varphi \in \delta_M^\vee$. If $M_A \neq \varnothing$ and φ is bounded below, then the minimum of φ on M satisfies (5-12). □

Corollary 5.10
Assume that $\varphi \in W^*$ is bounded below on non-empty polyhedron $M_A \subset U_\xi$. Then for any choice of $o \in U_\xi$

$$\min_{p \in M_A} \langle \varphi , p \rangle = \langle \varphi , o \rangle - \min_{y_\alpha} \sum_{\alpha \in A_{\mathrm{red}}} y_\alpha \cdot \langle \alpha , o \rangle , \tag{5-13}$$

where the minimum in R.H.S. is taken over all collections of non-negative real numbers y_α, $\alpha \in A_{\mathrm{red}}$, such that

$$D_\varphi = \sum_{\alpha \in A_{\mathrm{red}}} y_\alpha D_\alpha . \tag{5-14}$$

Proof. Let $\mu = \min_{p \in M_A} \langle \varphi , p \rangle$, $\mu^* = \min_{y_\alpha} \sum_{\alpha \in A_{\mathrm{red}}} y_\alpha \cdot \langle \alpha , o \rangle$. By (5-12) μ coincides with maximal m such that $m\xi = \varphi - \sum_{\alpha \in A_{\mathrm{red}}} y_\alpha \cdot \alpha$ for some $y_\alpha \in \mathbb{R}_{\geqslant 0}$. Evaluating both sides at o, we conclude that $\mu = \langle \varphi , o \rangle - \sum_{\alpha \in A_{\mathrm{red}}} y_\alpha \cdot \langle \alpha , o \rangle$ for some $y_\alpha \in \mathbb{R}_{\geqslant 0}$. Thus, $\mu \leqslant \langle \varphi , o \rangle - \mu^*$.

On the other side, if $m = \langle \varphi, o \rangle - \sum_{\alpha\in A_{\text{red}}} y_\alpha \cdot \langle \alpha, o \rangle$, then $m\xi = \langle \varphi, o \rangle \xi - \sum_{\alpha\in A_{\text{red}}} y_\alpha \cdot \langle \alpha, o \rangle \xi$ and $\varphi - m\xi = (\varphi - \langle \varphi, o \rangle \xi) + \sum_{\alpha\in A_{\text{red}}} y_\alpha \cdot \langle \alpha, o \rangle \xi$. Since φ is bounded below, there are $z_\alpha \geqslant 0$ such that $\varphi - \langle \varphi, o \rangle \xi = D_\varphi = \sum_{\alpha\in A_{\text{red}}} z_\alpha \cdot D_\alpha = \sum_{\alpha\in A_{\text{red}}} z_\alpha \cdot (\alpha - \langle \alpha, o \rangle \xi)$. Hence,

$$\varphi - m\xi = \xi \cdot \sum_{\alpha\in A_{\text{red}}} (z_\alpha + y_\alpha) + \sum_{\alpha\in A_{\text{red}}} z_\alpha \cdot \alpha \in \sigma_A .$$

Thus, m provides a lower boundary for φ on M, i.e. $\mu \geqslant m \geqslant \langle \varphi, o \rangle - \mu^*$. □

Corollary 5.11
The locus $\Gamma = \{q \in M \mid \langle \varphi, q \rangle = \min_{p\in M} \langle \varphi, q \rangle\}$ is a face of M. Its affine hull H_Γ coincides with the annihilator of all $\alpha \in A_{\text{red}}$ such that $y_\alpha \neq 0$ in some (and hence in any) collection of y_ν, $\nu \in A_{\text{red}}$, which satisfies (5-11) and produces the minimum in R.H.S. of (5-13).

Proof. Let $\max(m \mid \varphi - m\xi \in \sigma_A) = \min_{p\in M} \langle \varphi, p \rangle = m_*$. Then $\varphi - m^*\xi$ is a supporting functional for M and $\Gamma = H_{\varphi - m_*\xi} \cap M = \{p \in M \mid \langle \varphi, p \rangle = m_*\}$ is a face of M. By prop. 5.3, $\Gamma = H_\Gamma \cap M_A$, where $H_\Gamma = \bigcap_{\alpha\in A_\Gamma} H_\alpha$, $A_\Gamma = \{\alpha \in A_{\text{red}} \mid H_\alpha \supset \Gamma\}$. Since $\varphi - m_*\xi$ vanishes everywhere in Γ, in any decomposition $\varphi - m_*\xi = \sum_{\alpha\in A_{\text{red}}} y_\alpha \cdot \alpha$ non-zero coefficients y_α can appear only at $\alpha \in A_\varphi$, because each $\alpha \notin A_\Gamma$ is strictly positive somewhere in Γ. □

Corollary 5.12
Let R.H.S. of (5-13) achieves its minimum on some collection y_α and $A_\varphi = \{\alpha \in A \mid y_\alpha \neq 0\}$. Then the minimum in L.H.S. of (5-13) is achieved on $M \cap H_{A_\varphi} \neq \varnothing$, which is a face of M.

Proof. Let again $\max(m \mid \varphi - m\xi \in \sigma_A) = \min_{p\in M} \langle \varphi, p \rangle = \mu$. Then it follows from the proof of cor. 5.10 that $\varphi - \mu\xi = \sum_{\alpha\in A_f} y_\alpha \cdot \alpha$ and $M \cap H_{A_\varphi} \neq \varnothing$, because otherwise R.H.S. is strictly positive on M and L.H.S. vanishes along the face $H_{\varphi - \mu\xi} \cap M_A$. Thus, $0 = \min_M \varphi - \mu\xi$ is achieved exactly on the face $M \cap H_{A_\varphi}$. □

Home task problems к §5

Problem 5.1 (centre of mass). Show that a) for any collection of points $Q_1, Q_2, \dots, Q_m \in \mathbb{A}^2$ and any collection of constants[1] $\mu_1, \mu_2, \dots, \mu_m \in \Bbbk$ such that $\sum_{i=1}^{m} \mu_i = \mu \neq 0$ there exists a unique point $M \in \mathbb{A}^2$ such that $\mu_1 \overrightarrow{MQ}_1 + \mu_2 \overrightarrow{MQ}_2 + \cdots + \mu_m \overrightarrow{MQ}_m = 0$. b) for any point $P \in \mathbb{A}^2$ the point M equals $M = P + \sum_{i=1}^{m} \frac{\mu_i}{\mu} \cdot \overrightarrow{PQ}_i$.

Problem 5.2 (grouping masses). Let a finite collection of points Q_i with masses $\mu_i \in \Bbbk$ and a finite collection of points T_j with masses ν_j have centres of mass at points M and N respectively. Assume that all three sums $\sum \mu_i$, $\sum \nu_j$, $\sum \mu_i + \sum \nu_j$ are non-zeros. Show that centre of mass for the union of all points[2] Q_i and T_j coincides with the centre of mass of two points M, N, taken with the masses $\sum \mu_i$ and $\sum \nu_j$.

Problem 5.3. Give an example of a closed figure Φ with non-empty interior Φ° such that $\overline{\Phi^\circ} \neq \Phi$. Is it possible, if Φ is convex?

Problem 5.4. Give an example of a closed convex figure with a non-closed set of a) vertexes b) extremal points.

Problem 5.5 (Caratheodori's lemma). Show that each point of the convex hull of an arbitrary figure $\Phi \subset \mathbb{R}^n$ is a convex combination of at most $(n+1)$ points of Φ.

Problem 5.6 (Rhadon's lemma). Show that any finite set of $\geqslant (n+2)$ distinct points in $\mathbb{R}^n$ is a disjoint union of two non-empty subsets with intersecting convex hulls.

Problem 5.7 (Helly's theorem). For any set of closed convex figures in $\mathbb{R}^n$ such that at least one of the figures is compact and any $(n+1)$ figures have non-empty intersection show that the intersection of all the figures is non-empty.

Problem 5.8. Let a convex polyhedral cone $\sigma \in \mathbb{R}^3$ span the whole vector space. Show that σ and $\sigma^\vee$ have the same number of 1-dimensional edges. Give an example of polyhedral cone $\sigma \in \mathbb{R}^4$ such that σ and $\sigma^\vee$ have different numbers of 1-dimensional edges.

Problem 5.9. Show that a convex subspace η of a convex polyhedral cone σ is a face iff the following equivalence holds: $\forall\, v_1, v_2 \in \sigma\ v_1 + v_2 \in \eta \iff v_1, v_2 \in \eta$.

Problem 5.10. Show that any proper face τ of a convex polyhedral cone σ: a) is contained in some hyper-face[3] of σ b) coincides with the intersection of all hyper-faces of σ containing τ.

Problem 5.11. Let a convex polyhedral cone $\sigma \subsetneq V$ be generated by vectors $v_1, v_2, \dots, v_N$ that linearly span V, and $\dim V = n$. Prove that: a) the boundary $\partial\sigma$ is a union of all hyper-faces of σ

b) covectors $\xi_\tau \in V^*$ annihilating the hyper-surfaces $\sigma \subset \tau$ are contained in a finite set $M \subset V^*$ described as follows: list all the linearly independent collections of $(n-1)$ vectors v_ν; for each such collection find $\xi \in V^*$ that spans its annihilator; if for all generators v_i, $1 \leqslant i \leqslant N$,

[1] these constants are called «masses»
[2] «union» of coinciding points means adding their masses
[3] that is a face of codimension 1

$\langle \xi, v_i \rangle > 0$, then include ξ in M, else if for all v_i $\langle \xi, v_i \rangle < 0$, then include $-\xi$ in M, otherwise omit this ξ.

c) $\sigma = \bigcap_{\tau} H^+_{\xi_\tau}$, where $\tau \subset \sigma$ runs through the hyper-faces of σ.

Problem 5.12. Let $\xi \in \sigma^\vee$ and $\tau = \mathrm{Ann}(\xi) \cap \sigma$. Prove that $\tau^\vee = \{\zeta - \lambda\xi \mid \zeta \in \sigma^\vee,\ \lambda \geqslant 0\}$.

Problem 5.13. Let convex polyhedral cones σ_1 and σ_2 intersect each other precisely along a common face τ. Show that there exists $\xi \in \sigma_1^\vee \cap (-\sigma_2)^\vee$ such that $\tau = \sigma_1 \cap \mathrm{Ann}(\xi) = \sigma_2 \cap \mathrm{Ann}(\xi)$.

Problem 5.14 (Farkas lemma in terms of coordinates). Given a matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ and a column $b \in \mathbb{R}^m$ (of the same height as the matrix), prove that:

a) inequalities $Ax \leqslant b$ on a column $x \in \mathbb{R}^n$ are consistent iff for any non-negative row $y \in \mathbb{R}^{m*}_{\geqslant 0}$ equation $yA = 0$ implies inequality $yb \geqslant 0$

b) inequalities $Ax \leqslant b$ have a non-negative solution $x \in \mathbb{R}^n_{\geqslant 0}$ iff for any non-negative row $y \in \mathbb{R}^{m*}_{\geqslant 0}$ inequalities $yA \geqslant 0$ imply inequality $yb \geqslant 0$.

Problem 5.15 (Motzkin's decomposition). For any convex polyhedron $M \subset \mathbb{A}(V)$ construct a decomposition $M = \mathrm{conv}(p_1, p_2, \dots, p_m) + \delta_M = \{p + v \mid p \in \mathrm{conv}(p_1, p_2, \dots, p_m),\ v \in \delta_M\}$, where $\delta_M \subset V$ is the asymptotic cone[1] of M and $p_1, p_2, \dots, p_m \subset \mathbb{A}(V)$ is some finite collection of points. When M does not contain affine subspaces of positive dimension, show that $p_1, p_2, \dots, p_m \subset \mathbb{A}(V)$ can be taken to be the vertexes of M. Deduce from this that M is compact iff $\delta_M = 0$.

Problem 5.16. Show that any two vertexes of any convex polyhedron are connected by some pass formed from 1-dimensional edges.

Problem 5.17. Assume that a convex polyhedron $M \subset \mathbb{A}(V)$ does not contain affine subspaces of positive dimension. For each vertex $p \in M$ write $\sigma_p \subset V$ for a cone spanned by all the edges of M outgoing from p. Show that: a) $M_\infty = \bigcap_p \sigma_p$ b) $M \subset p + \sigma_p$ for any vertex p.

Problem 5.18. Let $M \subset \mathbb{A}(V)$ be a convex polyhedron with vertexes and covector $\xi \in V^*$ be bounded below on M. Show that:

a) there exist a vertex $p \in M$ such that $\langle \xi, x \rangle \geqslant \langle \xi, p \rangle$ for all $x \in M$

b) a vertex $p \in M$ satisfies the above property iff $\langle \xi, q \rangle \geqslant \langle xi, p \rangle$ for each edge $[p, q] \subset M$ outgoing from p (including those having q at infinity).

Problem 5.19 (regular polyhedrons). Given a polyhedron $M \subset \mathbb{R}^n$, *a group of M* is defined as a group of all bijections $M \xrightarrow{\sim} M$ induced by all euclidean linear automorphisms[2] of $\mathbb{R}^n$. Any sequence: vertex of M, edge of M outgoing from this vertex, 2-dimensional face of M outgoing from this edge, …, a hyper-face of M outgoing from theis $(n-1)$-dimensional face, M itself (all intermediate dimensions have to appear) is called *a flag* of M. A polyhedron M is called *regular*, if the group of M acts transitively on the flags of M. Given a regular polyhedron $P \subset \mathbb{R}^n$, we write $\ell = \ell(P)$ for the length of its edge, write $r = r(P)$ for the radius of its superscribed sphere, and put $\varrho = \varrho(P) \stackrel{\text{def}}{=} \ell^2/4r^2$. In all the problems below assume that a regular polyhedron $P \subset \mathbb{R}^n$ linearly spans the whole vector space.

a) (the star) Show that all vertexes of P joint with a given vertex $p \in P$ by an edge of P form a regular polyhedron in an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$. It is called *a star* of P and is denoted by $\mathrm{st}(P)$.

b) (the symbol) *Schläfli's symbol* of a regular polyhedron $P \subset \mathbb{R}^n$ is a collection of $(n-1)$

[1] see prop. 5.7 on p. 95

[2] we asume that $\mathbb{R}^n$ is equipped with the standard euclidean structure $|x|^2 = \sum x_i^2$

positive integers $\boldsymbol{\nu}(P) = \big(\nu_1(P), \nu_2(P), \dots, \nu_{n-1}(P)\big)$, defined inductively as follows: $\nu_1(P)$ equals the number of edges of 2-dimensional face of P and the rest sub-sequence

$$\big(\nu_2(P), \dots, \nu_{n-1}(P)\big) = \boldsymbol{\nu}\big(\mathrm{st}(P)\big)$$

is the Schläfli symbol of the star $\mathrm{st}(P)$. Find the symbols of the following regular polyhedrons: (1) dodecahedron in $\mathbb{R}^3$ (2) icosahedron in $\mathbb{R}^3$ (3) n-dimensional simplex (4) n-dimensional cube (5) n-dimensional cocube[1].

c) Express $\ell\big(\mathrm{st}(P)\big)$ through $\ell(P)$ and $\nu_1(P)$.

d) Show that $\varrho(P)$ depends only on the symbol of P and equals

$$\varrho(P) = 1 - \frac{\cos^2\big(\pi/\nu_1(P)\big)}{\varrho\big(\mathrm{st}(P)\big)}\,.$$

Problem 5.20 (duality). Let $P \subset \mathbb{R}^n$ be a regular polyhedron with the centre at the origin. a) Show that $P^* = \{\xi \in \mathbb{R}^{n*} \mid \xi(v) \geqslant -1 \ \ \forall\, v \in P\ \}$ a regular polyhedron with the centre at the origin. b) For each k construct a canonical bijection between k-dimensional spaces of P and $(n-k-1)$-dimensional faces of P^* reversing the inclusions of faces. c) Prove that a symbol of P^* is the symbol of P read from the right to the left.

Problem 5.21. Do the midpoints of edges in a) 3-dimensional b) 4-dimensional regular simplex form the vertexes of regular cocube?

Problem 5.22 (4-dimensional examples).

a) (octaplex) Let $I^4 = \{x \in \mathbb{R}^4 \mid \forall\, i\ |x_i| \leqslant 1\}$ be the standard cube and C^4 be the convex hull of endpoints of 8 vectors $\pm 2e_i$, where e_1, e_2, e_3, e_4 form the standard basis in $\mathbb{R}^4$. Convex hull of $I^4 \cup C^4$ is called *the octaplex* and is denoted by O^4. Find its symbol and compute the numbers of its vertexes, edges, 2- and 3-dimensional faces.

b) Write M for a convex hull of O^4 and all the points obtained from

$$\big(\pm\tau, \pm 1, \pm\tau^{-1}, 0\big)\,, \quad \text{where} \quad \tau^{\pm 1} = \frac{\sqrt{5} \pm 1}{2}\,,$$

by all even permutations of the coordinates. Show that M is regular polyhedron with symbol $(3, 3, 5)$ and compute the numbers of its vertexes, edges, 2- and 3-dimensional faces.

Problem 5.23 (clasification of regular polyhedrons). Show that the symbols of all regular polyhedrons $P \subset \mathbb{R}^n$ are contained in the following list:

a) (ν), where $\nu \geqslant 3$ is any positive integer, for $n = 2$

b) $(3, 3)$, $(3, 4)$, $(4, 3)$, $(3, 5)$, $(5, 3)$ for $n = 3$

c) $(3, 3, 3)$, $(3, 3, 4)$, $(4, 3, 3)$, $(3, 4, 3)$, $(3, 3, 5)$, $(5, 3, 3)$ for $n = 4$

d) $(3, \dots, 3)$, $(3, \dots, 3, 4)$, $(4, 3, \dots, 3)$ for $n \geqslant 5$

and for each symbol in the list there exists a unique up to dilatation regular polyhedron that has this symbol.

[1] that is, the convex hull of centres of the hyper-faces of the cube

Hints and answers for some exersices

Exrs. 1.5. Yes, it does.

Exrs. 1.9. First of all, check that $\dim U + \dim \operatorname{Ann} U = \dim V$ for any $U \subset V$. To this aim, pick up a basis $u_1, u_2, \dots, u_k \in U$, then complete it by some $w_1, w_2, \dots, w_m$ to a basis in V (thus, $\dim V = k+m$) and write $u_1^*, u_2^*, \dots, u_k^*, w_1^*, w_2^*, \dots, w_m^* \in V^*$ for the dual basis. Then $w_1^*, w_2^*, \dots, w_m^* \in \operatorname{Ann} U$, because for any $v = \sum x_i u_i \in U$ we have $w_\nu^*(v) = w_\nu^*\left(\sum x_i u_i\right) = \sum x_i w_\nu^*(u_i) = 0$. Since any covector $\varphi = \sum y_i u_i^* + \sum z_j w_j^* \in \operatorname{Ann}(U)$ has $y_i = \varphi(u_i) = 0$, covectors $w_1^*, w_2^*, \dots, w_m^*$ span $\operatorname{Ann}(U)$. Hence they form a basis and $\dim \operatorname{Ann}(U) = m = \dim V - \dim U$. Surely, the same equality holds for $U \subset V^*$, $\operatorname{Ann} U \subset V = V^{**}$.

Now, $U = \operatorname{Ann}\operatorname{Ann} U$, because $U \subset \operatorname{Ann}\operatorname{Ann}(U)$ by the definition and $\dim \operatorname{Ann}\operatorname{Ann} U = \dim U$ by just proven. Implication $U \subset W \Rightarrow \operatorname{Ann} U \supset \operatorname{Ann} W$ is obvious. If we apply it to $\operatorname{Ann} W$ as U and to $\operatorname{Ann} U$ as W and use the identities $\operatorname{Ann}\operatorname{Ann} W = W$, $\operatorname{Ann}\operatorname{Ann} U = U$, then we get the opposite implication $\operatorname{Ann} U \supset \operatorname{Ann} W \Rightarrow U \subset W$. The coincidence

$$\bigcap_\nu \operatorname{Ann}(U_\nu) = \operatorname{Ann}\left(\sum_\nu U_\nu\right), \tag{5-15}$$

is obvious: if a linear form annihilates each of U_ν's, then it annihilates their linear span and vice versa. Substituting $\operatorname{Ann} U_\nu$ instead of U_ν into (5-15), we get $\bigcap_\nu U_\nu = \operatorname{Ann}\left(\sum_\nu \operatorname{Ann} U_\nu\right)$. Passing to the annihilators of both sides, obtain $\operatorname{Ann}\left(\bigcap_\nu W_\nu\right) = \sum_\nu \operatorname{Ann}(W_\nu)$.

Exrs. 1.12. For any field $\mathbb{F}$ and any irreducible $g \in \mathbb{F}[x]$ one can pass to $\mathbb{F}' = \mathbb{F}[x]/(g) \supset \mathbb{F}$ in which g gets the root $\vartheta = [x]$ and becomes divisible by $(x - \vartheta)$ in $\mathbb{F}'[x]$. Applying this construction to non-linear irreducible factor g of f we strictly increase the number of linear factors in the factorization of f. Thus, after at most $\deg f - 1$ steps we factorize f completely.

Exrs. 1.13. If g.c.d.$(\alpha, \beta) = 1$, take $c = ab$ and verify that[1] $(ab)^k = 1 \Rightarrow \alpha | k \;\&\; \beta | k \Rightarrow (\alpha\beta) | k$. Hence, c has order $\alpha\beta$. If α and β are non-coprime, use prime factorization to construct *coprime* γ, δ such that $\gamma | \alpha$, $\delta | \beta$, and l.c.m.$(\alpha, \beta) = \gamma\delta$. Then show that elements $a' = a^{\alpha/\gamma}$ and $b' = b^{\beta/\delta}$ have orders δ and γ and put $c = a'b'$.

Exrs. 1.14. If $\varphi : \Bbbk \to \mathbb{F}$ takes $\varphi(a) = 0$ for some $a \neq 0$, then $\forall\, b \in \Bbbk$

$$\varphi(b) = \varphi\left(ba^{-1}a\right) = \varphi\left(ba^{-1}\right)\varphi(a) = 0\,.$$

Exrs. 1.15. R.H.S. consists of $q^n + q^{n-1} + \cdots + q + 1$ points. The cardinality of L.H.S. equals the number of non-zero vectors in $\mathbb{F}_q^{n+1}$ divided by the number of non-zero numbers in $\mathbb{F}_q$, that is $(q^{n+1} - 1)/(q - 1)$. We get the summation formula for geometric progression.

Exrs. 1.16. It is evident either from similar right triangles on fig. 1⋄3 on p. 12 or from identity $(s : 1) = (x_0 : x_1) = (1 : t)$.

Exrs. 1.18. $\binom{n+d}{d} - 1$.

Exrs. 1.19. In projective space any line does intersect any hyperplane, see prb. 1.12.

Exrs. 1.20. Let vector $v = u + w$ represent a point $p \in \mathbb{P}(V)$. Then $\ell = (u, w)$ passes through p and intersects K and L at u and w. Vice versa, if $v \in (a, b)$, where $a \in U$ and $b \in W$, then $v = \alpha a + \beta b$ and the uniqueness of the decomposition $v = u + w$ forces $\alpha a = u$ and $\beta b = w$. Hence $(ab) = \ell$.

[1] $x | y$ means «x divides y»

Exrs. 1.22. Let $L_1 = \mathbb{P}(U)$, $L_2 = \mathbb{P}(W)$, $p = \mathbb{P}(\Bbbk\cdot e)$. Then $V = W \oplus \Bbbk\cdot e$, because of $p \notin L_2$. Projection from p is a projectivisation of linear projection of V onto W along $\Bbbk\cdot e$. Since $p \notin L_1$, the restriction of this projection onto U has zero kernel. Thus, it produces linear projective isomorphism.

Exrs. 1.23. This is a particular case of exrs. 1.22.

Exrs. 1.24. Draw the cross-axix ℓ by joining $(a_1 b_2) \cap (b_1, a_2)$ and $(c_1 b_2) \cap (b_1, c_2)$). Then draw a line through b_1 and $\ell \cap (x, b_2)$. This line crosses ℓ_2 in $\varphi(x)$.

Exrs. 1.25. Let $[p_1, p_2, p_3, p_4] = [q_1, q_2, q_3, q_4]$. Assume that under homographies

$$\varphi_p : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1 \quad \text{and} \quad \varphi_q : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1$$

the pre-images of ∞, 0, 1 are p_1, p_2, p_3 and q_1, q_2, q_3. Then $\varphi_p(p_4) = \varphi_q(q_4)$ and $\varphi_q^{-1} \circ \varphi_p$ takes p_1, p_2, p_3, p_4 to q_1, q_2, q_3, q_4. Vice versa, let $\varphi_p : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1$ send p_1, p_2, p_3 to ∞, 0, 1 and φ_{qp} send p_1, p_2, p_3, p_4 to q_1, q_2, q_3, q_4. Then $\varphi_p \circ \varphi_{qp}^{-1}$ sends q_1, q_2, q_3, q_4 to ∞, 0, 1, $[p_1, p_2, p_3, p_4]$. Thus, $[p_1, p_2, p_3, p_4] = [q_1, q_2, q_3, q_4]$.

Exrs. 1.26. Since any two diagonals are non-perpendicular, the eigenvalues of an Euclidean isometry preserving each the diagonal could be only $(1, 1, 1)$, $(1, 1, -1)$, and $(-1, -1, -1)$.

Exrs. 1.27. Linear fractional mapping that takes $(p_2, p_1, p_3) \mapsto (\infty, 0, 1)$ is the initial one, which sent $(p_1, p_2, p_3) \mapsto (\infty, 0, 1)$, followed by one sending $(\infty, 0, 1) \mapsto (0, \infty, 1)$, which takes $\vartheta \mapsto 1/\vartheta$. Similarly, to permute (p_1, p_2, p_3) via cycles $(1, 3)$, $(2, 3)$, $(1, 2, 3)$, $(1, 3, 2)$ means to compose the initial mapping with one sending $(\infty, 0, 1)$ to $(1, 0, \infty)$, $(\infty, 1, 0)$, $(1, \infty, 0)$, $(0, 1, \infty)$ respectively, i.e. to take ϑ to $\vartheta/(\vartheta - 1)$, $1 - \vartheta$, $(\vartheta - 1)/\vartheta$, $1/(1 - \vartheta)$.

Exrs. 2.7. By the construction of the identification, a tangent line $T_p C$ to C at $p \in C$ consists of all pairs $\{p, r\}$, where r runs through C. Thus, $T_p C \cap T_q C = \{p, q\}$ as required. Given an involution $\sigma : \mathbb{P}_1 \xrightarrow{\sim} \mathbb{P}_1$ with fixed points p, q, for any pair of points a, b exchanged by σ we have $[a, b, p, q] = [b, a, p, q]$, because σ preserves cross-ratios. This forces $[a, b, p, q] = -1$. Vice versa, for given fixed $p, q \in \mathbb{P}_1$ and for any $a \in \mathbb{P}_1$ there exist a unique point b such that $[a, b, p, q] = -1$. Moreover, a and b are rational functions of each other (for fixed p, q). Thus, the rule $a \mapsto b$ defines an involution on $\mathbb{P}_1$ with fixed points p, q.

Exrs. 2.9. Linear span of columns of A has dimension 1. Take any basic vector v there. Then i th column of A equals $\xi_i \cdot v$ for some $\xi_i \in \Bbbk$. Thus $A = v \cdot \xi$.

Exrs. 2.10. Use *method of loci*: remove one of 4 given lines, describe the locus filled by lines crossing 3 remaining lines, then chose there those crossing the removed line.

Exrs. 2.14. Write $\beta(x, x)$ as $a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + \cdots$, where $a_1 a_2 a_3 \neq 0$, put $x_i = 0$ for $i > 3$, $x_3 = 1$, $x_2 = \vartheta_2$, $x_1 = \vartheta_1$, where $\vartheta_1, \vartheta_2 \in \mathbb{F}_p$ satisfy $a_1 \vartheta_1^2 + a_2 \vartheta_2^2 = -a_3$.

Exrs. 3.3. The same arguments like in exrs. 2.9 (see also n° 2.4.1 immediately after formula (exrs. 2.9), p. 39).

Exrs. 3.4. Choose some dual bases $u_1, u_2, \dots, u_n \in U$, $u_1^*, u_2^*, \dots, u_n^* \in U^*$ and a basis $w_1, w_2, \dots, w_m \in W$. Then mn decomposable tensors $u_i^* \otimes w_j$ form a basis in $U^* \otimes V$. The corresponding operators take

$$u_i^* \otimes w_j : u_k \mapsto \begin{cases} w_j & \text{for } k = i \\ 0 & \text{otherwise} \end{cases}.$$

Thus, matrix of $u_i^* \otimes w_j$ is the standard matrix unit having 1 in the crossing of j th row and i th column and zeros elsewhere.

Exrs. 3.5. For any linear mapping $f : V \to A$ the multiplication

$$V \times V \times \cdots \times V \to A\,,$$

which takes $(v_1, v_2, \dots, v_n)$ to their product $\varphi(v_1) \cdot \varphi(v_2) \cdot \cdots \cdot \varphi(v_n) \in A$, is multilinear. Hence, for each $n \in \mathbb{N}$ there exists a unique linear mapping $V^{\otimes n} \to A$ taking tensor multiplication to multiplication in A. Add them all together and get required algebra homomorphism $TV \to A$ extending f. Since any algebra homomorphism $TV \to A$ that extends f has to take $v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \varphi(v_1) \cdot \varphi(v_2) \cdot \cdots \cdot \varphi(v_n)$, it coincides with the extension just constructed. Uniqueness of free algebra is proved exactly like lemma 3.1 on p. 52.

Exrs. 3.6. Since decomposable tensors span $V^{*\otimes n}$ and equality

$$i_v\varphi(w_1, w_2, \dots, w_{n-1}) = \varphi(v,\, w_1, w_2, \dots, w_{n-1})$$

is linear in both v, φ, it is enough to check it for those φ which go to decomposable $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n$ under the isomorphism (3-11).

Exrs. 3.7. $\forall\, v, w \quad 0 = \varphi(\dots\,, (v+w),\, \dots\,, (v+w),\, \dots) = \varphi(\dots\,, v,\, \dots\,, w,\, \dots) + \varphi(\dots\,, w,\, \dots\,, v,\, \dots)$. Vice versa, if $\operatorname{char} \Bbbk \neq 2$, then $\varphi(\dots\,, v,\, \dots\,, v,\, \dots) = -\varphi(\dots\,, v,\, \dots\,, v,\, \dots)$ forces $\varphi(\dots\,, v,\, \dots\,, v,\, \dots) = 0$.

Exrs. 3.8. The same formal arguments as in lemma 3.1 on p. 52.

Exrs. 3.9. $\binom{n+d-1}{d-1}$ (the number of non-negative integer solutions $m_1, m_2, \dots, m_d$ of equation $m_1 + m_2 + \cdots + m_d = n$).

Exrs. 3.10. Each linear mapping $f : V \to A$ induces for each $n \in \mathbb{N}$ a symmetric multilinear map

$$V \times V \times \cdots \times V \to A\,,$$

taking $(v_1, v_2, \dots, v_n) \mapsto \prod \varphi(v_i)$. This gives a linear map $S^nV \to A$. All together they extend f to an algebra homomorphism $SV \to A$. Vice versa, each algebra homomorphism $SV \to A$ extending f takes $\prod v_i \to \prod \varphi(v_i)$ and coincides with the extension just constructed. Uniqueness is verified like in lemma 3.1 on p. 52.

Exrs. 3.11. The first follows from $0 = (v + w) \otimes (v + w) = v \otimes w + w \otimes v$, the second — from $v \otimes v + v \otimes v = 0$.

Exrs. 3.12. Modify appropriately the proof of prop. 3.1 on p. 58.

Exrs. 3.13. If $\dim V = d$ is even, the centre consists of all even polynomials[1]. If d is odd, the centre is spanned by even polynomials and the top degree component $\Lambda^d V$.

Exrs. 3.16. For any item in the sum its stabilizer in $\mathfrak{S}_n$ consists of $m_1!\; m_2! \cdots m_d!$ independent permutations of coinciding factors. Hence the length of $\mathfrak{S}_n$-orbit of this item is $\frac{n!}{m_1!\, m_2! \cdots m_d!}$.

Exrs. 3.17. Since multiplication map $g \mapsto g' = hg$ is bijective for each $h \in \mathfrak{S}_n$,

$$h\Big(\sum_{g\in\mathfrak{S}_n} g(t)\Big) = \sum_{g\in\mathfrak{S}_n} hg(t) = \sum_{g'\in\mathfrak{S}_n} g'(t)$$

$$h\Big(\sum_{g\in\mathfrak{S}_n} \operatorname{sgn}(g)\cdot g(t)\Big) = \operatorname{sgn}(h)\cdot \sum_{g\in\mathfrak{S}_n} \operatorname{sgn}(hg)\cdot hg(t) = \operatorname{sgn}(h)\cdot \sum_{g'\in\mathfrak{S}_n} \operatorname{sgn}(g)\cdot g'(t)\,.$$

[1]i.e. containing only the monomials of even degree

This implies that $h(\mathrm{sym}_n(t)) = \mathrm{sym}_n(t)$ and $h(\mathrm{alt}_n(t)) = \mathrm{sgn}(h)\cdot \mathrm{alt}_n(t)$ for all $h \in \mathfrak{S}_n$, i.e. (a) and (b) hold. Assertions (c) и (d) are obvious (both sums consist of $n!$ equal items). In (e) the separate summation over even and over odd permutation deals with the same items but taken with the opposite signs.

Exrs. 3.18. The first is straightforward computation. Then the equality $\mathrm{sym}_3 + \mathrm{alt}_3 + p = E$ implies that $\mathrm{im}(\mathrm{sym}_3) = \mathrm{Sym}^3(V)$, $\mathrm{im}(\mathrm{alt}_3) = \mathrm{Skew}^3(V)$ and $\mathrm{im}(p)$ span $V^{\otimes 3}$. Equalities $p \circ \mathrm{alt}_3 = \mathrm{alt}_3 \circ p = p \circ \mathrm{sym}_3 = \mathrm{sym}_3 \circ p = 0$ implies that each projector annihilates images of two other. This forces the sum of images to be a direct sum.

Exrs. 3.19. One has to check that $\mathrm{im}(p) = \{t \in V^{\otimes 3} \mid \left\langle (E + T + T^2)\xi\,,\, t \right\rangle = 0\ \forall\, \xi \in V^{*\otimes 3}\}$, where $\langle\, *\,,\, *\,\rangle$ means the complete contraction between $V^{*\otimes 3}$ and $V^{\otimes 3}$. Since $\langle\, g\xi\,,\, t\,\rangle = \left\langle\, \xi\,,\, g^{-1}t\,\right\rangle$ for all $g \in \mathfrak{S}_3$, $\xi \in V^{*\otimes 3}$, $t \in V^{\otimes 3}$, it is enough to check that $\mathrm{im}\, p$ coincides with the kernel of

$$\mathrm{Id}^{-1} + T^{-1} + T^{-2} = \mathrm{Id} + T^2 + T = 3(\mathrm{alt}_3 + \mathrm{sym}_3)\,.$$

But it is clear follows from solution of exrs. 3.18 that $\mathrm{alt}_3 + \mathrm{sym}_3$ projects $V^{\otimes 3}$ onto $\mathrm{Sym}^3 V \oplus \mathrm{Skew}^3 V$ along $\mathrm{im}(p)$.

Exrs. 3.21. Since the assertion is linear in v, f, g it can be checked only for $v = e_i$, $f = x_1^{m_1} \dots x_d^{m_d}$, $g = x_1^{k_1} \dots x_d^{k_d}$. This is straightforward from definitions.

Exrs. 3.22. It follows from the equality $\tilde{f}(v, x, \dots, x) = \frac{1}{n}\cdot \partial_v f(x)$, where $n = \deg f$.

Exrs. 3.24. This is similar to exrs. 3.21 on p. 66.

Exrs. 3.25. Choose a basis $e_1, e_2, \dots, e_m \in U$. If $\omega \notin \Lambda^m U$, there is a monomial e_J in the expansion of ω that does not contain some basic vector, say e_i. Then $e_i \wedge \omega \neq 0$, because it contains basic monomial $e_{i \sqcup J}$, which comes only as $e_i \wedge e_J$ that prevents its cancellation. Vice versa, if $\omega \in \Lambda^m U$, then $\omega = \lambda \cdot e_1 \wedge e_2 \wedge \cdots \wedge e_m$ and $\forall\, i\ e_i \wedge \omega = 0$, certainly. This forcs $u \wedge \omega = 0$ for all $u \in U$.

Exrs. 3.26. All non-trivial Plücker relations for $\omega = \sum_{i,j=1}^{4} a_{ij} e_i \wedge e_j$ are proportional to

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0\,,$$

which says that $\omega \wedge \omega = 0$.

Exrs. 4.2. (Comp. with general theory from n° 2.5 on p. 40.) The cone $C = P \cap T_p P$ consist of all lines passing through p and laying on P. On the other hand, it consists of all lines joining its vertex p with a smooth quadric $G = C \cap H$ cut out of C by any 3-dimensional hyperplane $H \subset T_p P$ complementary to p inside $T_p P \simeq \mathbb{P}_4$. Thus, any line on P passing through p has a form $(pp') = \pi_\alpha \cap \pi_\beta$, where $p' \in G$ and π_α, π_β are two planes spanned by p and two lines laying on the Segre quadric G and passing through p' (see fig. 4◇1 on p. 76).

Exrs. 4.4. If $\omega \in P$, then $Z = T_\omega P$ and $\omega = \mathfrak{u}(\ell)$ for some *lagrangian*[1] line $\ell \subset \mathbb{P}(V)$. This means that all lines in $\mathbb{P}_3$ intersecting the lagrangian line ℓ are lagrangian too and contradicts with non-degeneracy of Ω.

Exrs. 5.3. Since $\xi \mapsto 0$ the mapping is well defined. It is obviously injective.

[1]a line $(ab) \subset \mathbb{P}(V)$ is called *lagrangian* if $\Omega(u, w) = 0$ for all $u, w \in \ell$; this is equivalent to $\Omega(a, b) = 0$ and, by (4-5) and (4-3), to $\hat{q}\big(\omega, \mathfrak{u}(\ell)\big) = 0$

Exrs. 5.6. Clearly, $\overline{\Phi^\circ} \subset \Phi$. Let $p \in \Phi \smallsetminus \overline{\Phi^\circ}$ and $q \in \Phi^\circ$. Then p is an exterior point of $\overline{\Phi^\circ}$ and $[p, q]$ contains an internal point that is a boundary point of $\overline{\Phi^\circ}$. Show that this is impossible by joining p with points of some cubic neighbourhood of q contained in Φ°.

Exrs. 5.8. Each affine subspace is given by some system of linear non homogeneous equations $\left\langle \psi_i , x \right\rangle = c_i$. Any such equation is equivalent to a system of two linear non homogeneous inequalities $\left\langle \psi_i , x \right\rangle \geqslant c_i$ and $\left\langle \psi_i , x \right\rangle \leqslant c_i$. Empty set and the whole space are given by inequalities $1 \leqslant 0$ and $1 \geqslant 0$.

Exrs. 5.11. It follows from the definitions of n° 5.4.1 that $-\sigma^\vee_{R^\vee_\Gamma} = \{ v \in V \mid \forall \psi \in \sigma_{R^\vee_\Gamma} \langle \psi , v \rangle \leqslant 0 \}$. For $v \in \sigma_R$ and $\psi \in \sigma^\vee_R$ the condition $\langle \psi , v \rangle \leqslant 0$ means the equality $\langle \psi , v \rangle = 0$. Thus, $\sigma \cap \left(-\sigma^\vee_{R^\vee_\Gamma} \right) = \sigma \cap H_\Gamma = \Gamma$.