## Classical derived functors

In this section[1] we will work in category of modules over a commutative ring $K$ although almost all the constructions without any changes can be applied in much more general categories with tensor products, for example for modules over non-commutative rings.

<u>Tensor products.</u> Let $M$, $N$ be two $K$-modules. Then their *tensor product* over $K$ is defined as an universal $K$-bilinear map $M \times N \xrightarrow{\tau} M \otimes N$ such that any $K$-bilinear map $M \times N \xrightarrow{\varphi} W$ is uniquely decomposed as $\varphi = \widetilde{\varphi} \circ \tau$, where $M \otimes N \xrightarrow{\widetilde{\varphi}} W$ is $K$-linear. So, the definition is the same as for the vector spaces if $K$ would be a field; in particular, any two tensor products, if exist, are *canonically* isomorphic and a tensor product of free modules based by sets $\{e_m\}$, $\{e_n\}$ is a free module based by $\{e_m \otimes e_n\}$.

Exercice 6.1. Check that Claim 2.3 holds for any $K$-modules and Claim 2.4 holds for any free $K$-modules over an arbitrary commutative ring.

Exercice 6.2. Check that $K^\nu \otimes N = N^\nu$ for any $K$-module $N$ (here $\nu$-th power means $\nu$-typle direct sum with itself).

To prove the existence in general case, consider some presentations of $M$, $N$ via generators and relations:

$$K^{\mu'} \xrightarrow{\delta_1^M} K^{\mu} \xrightarrow{\delta_0^M} M \longrightarrow 0 \qquad\qquad K^{\nu'} \xrightarrow{\delta_1^N} K^{\nu} \xrightarrow{\delta_0^N} N \longrightarrow 0$$

Here $\mu$ is a set of $K$-linear generators for $M$, $K^\mu$ is free $K$-module, whose basic vectors $e_m$ are labelled by $m \in \mu \subset M$, the map $\delta_0^M$ takes $e_m \in K^\mu$ to the corresponding vector $m \in M$. Similarly, $\mu'$ is a set of relations between the generators $m$, i.e. free generators $e_{m'} \in K^{\mu'}$ stay in 1–1 correspondence with linear combinations $\varrho_{m'} = \sum\limits_m \lambda_{m'}^m \cdot e_m \in K^\mu$, which go to zero under $\delta_0^M$ (i.e. become $0 \in M$ after replacing each $e_m$ by $m$) and span $\ker \delta_0^M$ over $K$. Clearly, $\delta_1^M$ takes $e_{m'} \longmapsto \varrho_{m'}$ and, of course, we suppose the same conventions for $\nu$, $\nu'$, $\delta_0^N$, $\delta_1^N$, and $N$ as well.

<u>Claim 6.3</u>. *$M \otimes N = K^\mu \otimes K^\nu / (K^\mu \otimes \ker \delta_0^N + \ker \delta_0^M \otimes K^\nu)$ with $\tau : (m,n) \longmapsto e_m \otimes e_n \; \big(\mathrm{mod}\,(K^\mu \otimes \ker \delta_0^N + \ker \delta_0^N \otimes K^\nu)\big)$ gives the tensor product of $M$, $N$ (i.e. $M \otimes N$ is spanned by the generators $e_m \otimes e_n$, where $m \in \mu$, $n \in \nu$, modulo the relations spanned by $e_m \otimes \varrho_{n'}$ and $\varrho_{m'} \otimes e_n$, where $m' \in \mu'$, $n' \in \nu'$).*

Proof. Since $\tau$ sends any relation between the generators $\{m\}$ coupled with any $y \in N$ to $\ker \delta_0^M \otimes K^\nu$ and, similarly, any $x \in M$ coupled with any relation between the generators $\{n\}$ — to $K^\mu \otimes \ker \delta_0^N$, then $M \times N \xrightarrow{\tau} M \otimes N$ is well defined. Further, any bilinear map $M \times N \xrightarrow{\varphi} W$ forces $\widetilde{\varphi}$ to send $e_m \otimes e_n \longmapsto \varphi(m,n)$. Again, since $\varphi$ was well defined, this rule gives well defined map $M \times N \xrightarrow{\widetilde{\varphi}} W$ factorizing $\varphi$ trough $\tau$. $\square$

In other words, there is a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
K^{\mu'} \otimes K^{\nu'} \arrow[r, "\delta_1^M \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \delta_1^N"] & K^{\mu} \otimes K^{\nu'} \arrow[r, "\delta_0^M \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \delta_1^N"] & M \otimes K^{\nu'} \arrow[r] \arrow[d, "\mathrm{Id} \otimes \delta_1^N"] & 0 \\
K^{\mu'} \otimes K^{\nu} \arrow[r, "\delta_1^M \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \delta_0^N"] & K^{\mu} \otimes K^{\nu} \arrow[r, "\delta_0^M \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes \delta_0^N"] & M \otimes K^{\nu} \arrow[r] \arrow[d, "\mathrm{Id} \otimes \delta_0^N"] & 0 \\
K^{\mu'} \otimes N \arrow[r, "\delta_1^M \otimes \mathrm{Id}"] \arrow[d] & K^{\mu} \otimes N \arrow[r, "\delta_0^M \otimes \mathrm{Id}"] \arrow[d] & M \otimes N \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd} \tag{34}$$

and $M \otimes N = K^\mu \otimes K^\nu / \Big( K^\mu \otimes \delta_1(K^{\nu'}) + \delta_1(K^{\mu'}) \otimes K^\nu \Big)$.

Exercice 6.4. Show that $M \otimes N$ also can be constructed as the factor module of a free module spanned by all pairs $(m,n) \in M \times N$ through a submodule spanned by all linear combinations of form

$$((x_1 m_1 + x_2 m_2), n) - x_1(m_1, n) - x_2(m_2, n)\ , \quad (m, (x_1 n_1 + x_2 n_2)) - x_1(m, n_1) - x_2(m, n_2)\ , \tag{35}$$

[1] Notes on this part of the course are still non complete, unfortunately…

where $m, m_i \in M$, $n, n_i \in N$, $x_i \in K$, and verify that the relations (35) imply also the relations $x(m \otimes n) = (xm) \otimes n = m \otimes (xn)$, where $m \otimes n$ is the class of $(m, n)$ modulo the relations (35).

Exercice 6.5. Let $F = K^\nu$ be a free $K$-module. Check that each exact triple of modules

$$0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$$

produces an exact triple of the tensor products: $0 \longrightarrow L \otimes F \xrightarrow{\varphi\otimes\mathrm{Id}} M \otimes F \xrightarrow{\psi\otimes\mathrm{Id}} N \otimes F \longrightarrow 0$.

Claim 6.6. *Tensoring an exact triple* $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ *by an arbitrary (possibly non free) $K$-module $R$, one gets an exact triple*

$$L \otimes R \xrightarrow{\varphi\otimes\mathrm{Id}} M \otimes R \xrightarrow{\psi\otimes\mathrm{Id}} N \otimes R \longrightarrow 0 . \tag{36}$$

*whose leftmost arrow $\varphi \otimes \mathrm{Id}$, however, can be non injective in general.*

Proof. Since $\psi \otimes \mathrm{Id}(m \otimes r) = n \otimes r$ as soon $\psi(m) = n$, the right arrow $\psi \otimes \mathrm{Id}$ in (36) is clearly surjective. Evidently, $(\psi \otimes \mathrm{Id})\circ(\varphi \otimes \mathrm{Id}) = (\psi\circ\varphi) \otimes \mathrm{Id} = 0$. So, it remains to check the inclusion $\ker(\psi \otimes \mathrm{Id}) \subset \mathrm{im}\,(\varphi \otimes \mathrm{Id})$. To this aim, note that prescription $n \otimes r \longmapsto m \otimes r$, where $m \in M$ is such that $\psi(m) = n$, gives a well defined map

$$N \otimes R \xrightarrow{\sigma} \frac{M \otimes R}{\mathrm{im}\,(\varphi \otimes \mathrm{Id})} ,$$

which does not depend on a choice of $m \in \psi^{-1}(n) = m + \mathrm{im}\,(\varphi)$ and satisfies $\sigma\circ(\psi \otimes \mathrm{Id}) = \mathrm{Id}_{\frac{M\otimes R}{\mathrm{im}\,(\varphi\otimes\mathrm{Id})}}$. This implies that $\psi \otimes \mathrm{Id}$ does not annihilate non zero classes in $(M \otimes R)/\mathrm{im}\,(\varphi \otimes \mathrm{Id})$, i. e. $\ker(\psi \otimes \mathrm{Id}) \subset \mathrm{im}\,(\varphi \otimes \mathrm{Id})$ as required.

What about possible non-injectivity of $\varphi \otimes \mathrm{Id}$, let us put $K = \mathbb{Z}$ and multiply an exact triple of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{z\mapsto 2z} \mathbb{Z} \longrightarrow (\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

by $\mathbb{Z}/2\mathbb{Z}$. Then we get the triple $(\mathbb{Z}/2\mathbb{Z}) \xrightarrow{0} (\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\mathrm{Id}} (\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$. $\square$

Exercice 6.7. Show that the tensor product over $\mathbb{Z}$ of torsion $\mathbb{Z}$-modules $M = \mathbb{Z}/m\mathbb{Z}$ and $N = \mathbb{Z}/n\mathbb{Z}$ equals $\mathbb{Z}/d\mathbb{Z}$, where $d = \textsc{g.c.d.}(m, n)$ (in particular, $M \otimes N = 0$ as soon $m$, $n$ are coprime).

Exercice 6.8. Show that for any ideal $I \subset K$ the tensor product of $K$-modules $I$ and $K/I$ coincides as a $K$-module with $I/I^2$.

Projective resolutions. Each $K$-module $M$ is a factor module of some free $K$-module $F$. For example, one can take $F = K^M$ to be based by $e_m$, where $m$ runs through $M$; then the natural map $K^M \twoheadrightarrow M$, which takes $e_m \longmapsto m$, is clearly surjective. Given such a presentation $F_0 \overset{\delta_0}{\twoheadrightarrow} M$ with free $F_0$, we can consider a similar representation $F_1 \overset{\delta_1}{\twoheadrightarrow} \ker \delta_0$ for its kernel e. t. c. . This leads to exact sequence (possibly non bounded from the left)

$$\cdots \xrightarrow{\delta_3} F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

called *a free resolution* of $M$. Surjectivity of maps $F_1 \overset{\delta_i}{\twoheadrightarrow} \ker(\delta_{i-1})$ for $i = 0, 1, 2, \dots$ means that free generators of $F_0$ stay in 1–1 correspondence with some system of generators for $M$; free generators of $F_1$ stay in 1–1 correspondence with with some system of linear relations between these generators that linearly generates the whole set of linear relations; free generators of $F_2$ stay in 1–1 correspondence with some generating linear relations between these relations e. t. c. After D. Hilbert, basic elements of $F_i$ are called *syzygies* of $i$-th order for $M$. In fact, *a complete description* of a given $K$-module $M$ supposes knowledge of all its syzygies.

Of course, each module has a lot of different free resolutions. To compare them, it is convenient to work with slightly more general *projective* modules instead of free ones. $K$-module $P$ is called *projective* if it satisfies the following conditions:

1) $\exists$ $K$-module $Q$ such that $P \oplus Q$ is free;

2) any surjection $F \overset{\pi}{\twoheadrightarrow} P$, where $F$ is a free $K$-module, admits a section $P \overset{\psi}{\hookrightarrow}$ such that $\pi\psi = \mathrm{Id}_P$;

3) any $K$-module homomorphism $P \xrightarrow{\varphi} N$ can be lifted along any surjection $M \overset{\pi}{\twoheadrightarrow} N$, i. e. $\exists$ $P \xrightarrow{\psi} M$ such that $\pi\psi = \varphi$;

4) functor $h^P : N \longmapsto \mathrm{Hom}(P, N)$ takes any exact triple of modules $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ to an exact triple $0 \longrightarrow \mathrm{Hom}(P, L) \longrightarrow \mathrm{Hom}(P, M) \longrightarrow \mathrm{Hom}(P, N) \longrightarrow 0$

5) functor $h^P$ takes any exact sequence of modules to an exact sequence.

Lemma 6.9. *These conditions are pairwise equivalent.*

Proof. (4) ⇔ (5), because any long exact sequence $\cdots \longrightarrow L_{i-1} \xrightarrow{\varphi_{i-1}} L_i \xrightarrow{\varphi_i} L_{i+1} \longrightarrow \cdots$ is glued from exact triples $\ldots, 0 \longrightarrow \ker(\varphi_{i-1}) \longrightarrow L_{i-1} \longrightarrow \operatorname{im}(\varphi_{i-1}) \longrightarrow 0,\ 0 \longrightarrow \ker(\varphi_i) \longrightarrow L_i \longrightarrow \operatorname{im}(\varphi_i) \longrightarrow 0, \ldots$ in which $\operatorname{im}(\varphi_{i-1}) = \ker(\varphi_i)$.

(3) ⇔ (4): if we apply $\operatorname{Hom}(P, *)$ with an arbitrary $P$ to an exact triple $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ then anyway we get an exact sequence

$$0 \longrightarrow \operatorname{Hom}(P, L) \longrightarrow \operatorname{Hom}(P, M) \longrightarrow \operatorname{Hom}(P, N) \tag{37}$$

whose rightmost arrow, however, may be non surjective in general.

Exercice 6.10. Verify this using the straightforward diagram searching.

So, (4) is equivalent to the surjectivity of map $\operatorname{Hom}(P, M) \xrightarrow{\psi \mapsto \pi\psi} \operatorname{Hom}(P, N)$ induced by any surjection $M \overset{\pi}{\twoheadrightarrow} N$.

(3) ⇒ (2) via applying (3) to $M = F$, $N = P$. and $\varphi = \mathrm{Id}_P$.

(2) ⇒ (1) by taking any epimorphism $F \overset{\pi}{\twoheadrightarrow} P$ with free $F$ and decomposing $F = \ker \pi \oplus \operatorname{im} \psi$. Indeed, $\forall\, f \in F$ we have $f = (f - \psi\pi f) + \psi\pi f$ but $\pi\psi = \mathrm{Id}_P$ implies that: $\pi(f - \psi\pi f) = \pi f - \mathrm{Id}_P \pi f = 0$; $\ker \pi \cap \operatorname{im} \psi = 0$; $\operatorname{im} \psi \simeq P$.

To show that (1) ⇒ (3) consider a free module $F = P \oplus Q$, write $\{e_\nu\}$ for its free generators and $\{e'_\nu\}$ for their projections onto $P$ along $Q$. Then, given surjective $M \overset{\pi}{\twoheadrightarrow} N$ and any $P \xrightarrow{\varphi} N$, define a map $F \xrightarrow{\psi} M$ by sending $e_i$ to any $m \in M$ such that $\pi(m) = \varphi(e'_i)$. Restriction of $\psi$ onto $P$ gives a required lifting. □

A complex of projective modules $\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$ is called *a projective resolution* of a module $M$, if it is exact everywhere except for $P_0$ where $H_0(P_\bullet) = M$. Equivalently, this means that there is an exact sequence $\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$. In particular, each free resolution of $M$ is a projective resolution as well.

Lemma 6.11. *Let $P_\bullet$ and $Q_\bullet$ be projective resolutions of modules $M$ and $N$ respectively. Then any morphism $M \xrightarrow{\varphi} N$ can be lifted to a morphism of complexes $P_\bullet \xrightarrow{\psi} Q_\bullet$, which induces $\varphi$ on homology. Any two such liftings $\psi$, $\psi'$ are homotopy equivalent.*

Proof. We need to construct a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_3^P"] & P_2 \arrow[r, "\partial_2^P"] \arrow[d, "\psi_2"] & P_1 \arrow[r, "\partial_1^P"] \arrow[d, "\psi_1"] & P_0 \arrow[r, "\partial_0^P"] \arrow[d, "\psi_0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r, "\partial_3^Q"] & Q_2 \arrow[r, "\partial_2"] & Q_1 \arrow[r, "\partial_1^Q"] & Q_0 \arrow[r, "\partial_0^Q"] & N \arrow[r] & 0
\end{tikzcd}$$

by adding vertical arrows $\psi_i$ to the given exact rows and rightmost vertical arrow $\varphi$. Let us use induction on $i$ starting with $\psi_{-1} = \varphi$. If $\psi_{i-1}$ is already constructed, then we have a map $P_i \xrightarrow{\psi_{i-1}\partial_i^P} \ker \partial_{i-1}^Q$ and surjection $Q_i \overset{\partial_i^Q}{\twoheadrightarrow} \ker \partial_{i-1}^Q$. Since $P_i$ is projective, there is a lifting $P_i \xrightarrow{\psi_i} Q_i$ such that $\partial_i^Q \psi_i = \psi_{i-1}\partial_i^P$.

Now for any two liftings $\psi_i$, $\psi'_i$ we need to construct a series of diagonal arrows

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_3^P"] & P_2 \arrow[r, "\partial_2^P"] \arrow[d, "\Delta\psi_2"] \arrow[dl, "\gamma_2"'] & P_1 \arrow[r, "\partial_1^P"] \arrow[d, "\Delta\psi_1"] \arrow[dl, "\gamma_1"'] & P_0 \arrow[r, "\partial_0^P"] \arrow[d, "\Delta\psi_0"] \arrow[dl, "\gamma_0"'] & M \arrow[r] \arrow[d, "\Delta\varphi=0"] \arrow[dl, "\gamma_{-1}"'] & 0 \\
\cdots \arrow[r, "\partial_3^Q"] & Q_2 \arrow[r, "\partial_2"] & Q_1 \arrow[r, "\partial_1^Q"] & Q_0 \arrow[r, "\partial_0^Q"] & N \arrow[r] & 0
\end{tikzcd}$$

where $\Delta\psi_i \overset{\text{def}}{=} \psi_i - \psi'_i = \partial_{i+1}^Q \gamma_i + \gamma_{i-1}\partial_i^P$. Let us again use induction over $i$ starting with $\gamma_{-1} = 0$. If $\gamma_{i-1}$ is constructed, then the map $\Delta\psi_i - \gamma_{i-1}\partial_i^P$ takes $P_i$ to $\ker \partial_i^Q$, because of

$$\partial_i^Q(\Delta\psi_i - \gamma_{i-1}\partial_i^P) = (\Delta\psi_{i-1} - \partial_i^Q\gamma_{i-1})\partial_i^P = \gamma_{i-2}\partial_{i-1}^P\partial_i^P = 0\,.$$

So, it can be lifted along the surjection $Q_{i+1} \overset{\partial_{i+1}^Q}{\twoheadrightarrow} \ker \partial_i^Q$, i. e. there exists $P_i \xrightarrow{\gamma_i} Q_{i+1}$ such that $\partial_{i+1}^Q\gamma_i = \Delta\psi_i - \gamma_{i-1}\partial_i^P$ as required. □

Corollary 6.12. *Any two projective resolutions $P_\bullet$, $Q_\bullet$ for a given module $M$ are homotopy equivalent*[2]*, i. e. there are two maps of complexes $P_\bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Q_\bullet$ such that $\alpha\beta$ is homotopy equivalent to $\mathrm{Id}_Q$ and $\beta\alpha$ is homotopy equivalent to $\mathrm{Id}_P$.*

Proof. Let the maps $\alpha$, $\beta$ lift the identity map $\mathrm{Id}_M$ via Lemma 6.11. Then $\alpha\beta$ and $\beta\alpha$ also lift $\mathrm{Id}_M$ and again by Lemma 6.11 should be homotopy equivalent to $\mathrm{Id}_Q$ and $\mathrm{Id}_P$ respectively. □

[2]that is isomorphic in the homotopy category of modules

Example 6.13: Cartan – Eilenberg resolution. Given an exact triple of modules, let us fix some projective resolutions for two extremes and put direct sums of the corresponding projective modules in the middle:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & P_2 \arrow[r, hook] \arrow[d, "\delta_2"'] & P_2 \oplus Q_2 \arrow[r, two heads] \arrow[d, dotted, "\partial_2"] & Q_2 \arrow[d, "d_2"] & \\
 & P_1 \arrow[r, hook] \arrow[d, "\delta_1"'] & P_1 \oplus Q_1 \arrow[r, two heads] \arrow[d, dotted, "\partial_1"] & Q_1 \arrow[d, "d_1"] & \\
 & P_0 \arrow[r, hook] \arrow[d, "\delta_0"'] & P_0 \oplus Q_0 \arrow[r, two heads] \arrow[d, dotted, "\partial_0"] & Q_0 \arrow[d, "d_0"] & \\
0 \arrow[r] & L \arrow[r, "\varphi"] & M \arrow[r, "\psi"] & N \arrow[r] & 0
\end{tikzcd}$$

Then the middle terms of the horizontal splitted triples can be arranged into projective resolution for $M$ as follows. On $m$-th floor choose a map $Q_i \xrightarrow{\eta_i} P_{i-1} \oplus Q_{i-1}$, which lifts the differential $Q_i \xrightarrow{d_i} Q_{i-1}$ along the surjection $P_{i-1} \oplus Q_{i-1} \twoheadrightarrow Q_{i-1}$, and define $\partial_i$ by prescription $(p,q) \longmapsto (\delta_i(p), 0) + \eta_i(q)$.

Exercice 6.14. Check that the diagram constructed by this rule is commutative and contains a projective resolution for $M$ in the middle column.

Now, for any complex of modules $\cdots \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \longrightarrow 0$, we can construct a bicomplex $P_{p,q}$ (situated in degrees $p, q \geqslant 0$) with the following properties:

- $p$-th column of $P_{p,q}$ is a projective resolution for $M_p$;
- homologies $H^{\text{hor}}_{p,q}$, of the horizontal row-complexes in $P_{p,q}$, are projective modules;
- vertical differentials of $P_{p,q}$ arrange this horizontal homologies into projective resolutions for the homologies of $M$, i. e. in $p$-th column we get an induced exact sequence:

$$\cdots \longrightarrow H^{\text{hor}}_{p,2} \longrightarrow H^{\text{hor}}_{p,1} \longrightarrow H^{\text{hor}}_{p,0} \longrightarrow H_p(M_\bullet) \longrightarrow 0\,.$$

This bicomplex is called *projective Cartan – Eileberg resolution* for the initial complex $M$. To construct it, let us break $M_\bullet$ into exact triples:

$$\begin{array}{c}
\cdots\cdots\cdots\cdots\cdots\cdots \\
0 \longrightarrow \ker(\partial_{i+1}) \longrightarrow M_{i+1} \longrightarrow \operatorname{im}(\partial_{i+1}) \longrightarrow 0 \\
0 \longrightarrow \operatorname{im}(\partial_{i+1}) \longrightarrow \ker(\partial_i) \longrightarrow H_i(M_\bullet) \longrightarrow 0 \\
0 \longrightarrow \ker(\partial_i) \longrightarrow M_i \longrightarrow \operatorname{im}(\partial_i) \longrightarrow 0 \\
0 \longrightarrow \operatorname{im}(\partial_i) \longrightarrow \ker(\partial_{i-1}) \longrightarrow H_{i-1}(M_\bullet) \longrightarrow 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}$$

and for each $p \geqslant 0$ choose some projective resolutions $Q^p_\bullet \twoheadrightarrow \operatorname{im}(\partial_p)$ and $R^p_\bullet \twoheadrightarrow H_p(M_\bullet)$. Then the above construction produces projective resolutions $\left(Q^{p+1}_\bullet \oplus R^p_\bullet\right) \twoheadrightarrow \ker(\partial_p)$ and $\left(Q^{p+1}_\bullet \oplus R^p_\bullet \oplus Q^p_\bullet\right) \twoheadrightarrow M_p$. The last can be glued back into two-dimensional commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\delta_3"] & Q^3_\bullet \oplus R^2_\bullet \oplus Q^2_\bullet \arrow[r, "\delta_2"] \arrow[d] & Q^2_\bullet \oplus R^1_\bullet \oplus Q^1_\bullet \arrow[r, "\delta_1"] \arrow[d] & Q^1_\bullet \oplus R^0_\bullet \oplus Q^0_\bullet \arrow[r, "\delta_0"] \arrow[d] & 0 \\
\cdots \arrow[r, "\partial_3"] & M_2 \arrow[r, "\partial_2"] & M_1 \arrow[r, "\partial_1"] & M_0 \arrow[r] & 0
\end{tikzcd}$$

whose $p$-th column is the projective resolutions for $M_p$ just constructed and the horizontal differential at $(p,q)$-cell

$$Q^{p+1}_\bullet \oplus R^p_\bullet \oplus Q^q_\bullet \xrightarrow{\delta} Q^p_\bullet \oplus R^{p-1}_\bullet \oplus Q^{p-1}_\bullet$$

acts identically on $Q^p_q$ and annihilate two other summands. Now, taking $P_{p,q} = Q^{p+1}_q \oplus R^p_q \oplus Q^p_q$ and multiplying the vertical differential at $(p,q)$-cell by $(-1)^{p+q}$, we get the required bicomplex.

Exercice 6.15. Verify the properties listed before.

<u>FUNCTOR *Tor*.</u> We have seen that the structure of $M \otimes N$ becomes quite unclear for non free $M$ and $N$. Regular technology for its complete description requires a complete description of modules $M$, $N$ (i.e. their free resolution) and goes as follows. Take a free resolution $P_\bullet \twoheadrightarrow M$. Applying $* \otimes N$ to the complete

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_0} P_0 \longrightarrow 0$$

we get a complex

$$\cdots \xrightarrow{\partial_3 \otimes \mathrm{Id}_N} P_2 \otimes N \xrightarrow{\partial_2 \otimes \mathrm{Id}_N} P_1 \otimes N \xrightarrow{\partial_1 \otimes \mathrm{Id}_N} P_0 \otimes N \longrightarrow 0$$

Its $p$-th homology $Tor_p(M,N) \stackrel{\text{def}}{=} H_p(P_\bullet \otimes N)$ is called *$p$-th torsion product* of $M$, $N$. These products have a lot of remarkable properties.

First of all, they do not depend on a choice of free resolution; moreover *any projective* resolution can be used to compute them, because change of a projective resolution leads to homotopy equivalent complex with isomorphic homologies. Secondary, $Tor_p(P,N) = 0$ for all $p \geqslant 1$ and all $N$ as soon as $P$ is projective, because there is 1-term projective resolution $P_0 = P$ for $P$. Finally, since the functor $* \otimes N$ takes exact triple

$$0 \longrightarrow \operatorname{im} \partial_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

to exact sequence $\operatorname{im} \partial_1 \otimes \mathrm{Id}_N \longrightarrow P_0 \otimes N \longrightarrow M \otimes N \longrightarrow 0$ we have

$$Tor_0(M,N) = \frac{P_0 \otimes N}{\operatorname{im} \partial_1 \otimes \mathrm{Id}_N} = M \otimes N$$

By this reason, higher $Tor$'s also are called *the left derived functors* for the functor $* \otimes N$.

Exercice 6.16. Compute $Tor_\bullet(\mathbb{Z}/n\mathbb{Z}\,,\,\mathbb{Z}/m\mathbb{Z})$ in the category of $\mathbb{Z}$-modules.

HINT. Show that $Tor$'s are biadditive w.r.t. the direct sums; after that it is enough to consider two cases: G.C.D.$(m,n) = 1$ and $m = p^\mu$, $n = p^\nu$; tensoring by $\mathbb{Z}/m\mathbb{Z}$ the free resolution $\mathbb{Z} \xrightarrow{z \mapsto nz} \mathbb{Z}$ for $\mathbb{Z}/n\mathbb{Z}$, we see at once that all $Tor$'s vanish in the first case and are equal to ... in the second.

More important remark is that the same homologies can be described also either as $H_p(M \otimes Q_\bullet)$, where $Q_\bullet \twoheadrightarrow N$ is any projective resolution for $M$, or as $H_p(Tot(P_\bullet \otimes Q_\bullet))$, where $Tot(P_\bullet \otimes Q_\bullet)$ is the total complex of the bicomplex obtained by tensoring any two projective resolutions $P_\bullet \twoheadrightarrow M$, $Q_\bullet \twoheadrightarrow N$. To see this, let as arrange $P_p \otimes Q_q$ in the cells of the first coordinate octant with $p$-direction being horizontal and consider two spectral sequences[3] computing $H_{p+q}(Tot(P_\bullet \otimes Q_\bullet))$.

The first has $'E^2_{pq} = H^{\text{ver}}_q(H^{\text{hor}}_p(P_\bullet \otimes Q_\bullet))$. Since each row $P_\bullet \otimes Q_q$ is the result of tensoring by $Q_q$ the projective resolution $P_\bullet \twoheadrightarrow M$, the horizontal homologies

$$H^{\text{hor}}_p(P_\bullet \otimes Q_q) = Tor_p(M, Q_q)$$

vanish everywhere except for $Tor_0(M,Q_p) = M \otimes Q_q$, because each $Q_q$ is projective. This means that the horizontal homologies are situated along the zero column and vertical differential arranges them into complex

$$\cdots \longrightarrow M \otimes Q_2 \longrightarrow M \otimes Q_1 \longrightarrow M \otimes Q_0 \longrightarrow 0$$

obtained by tensoring the projective resolution $Q_\bullet \twoheadrightarrow N$ by $M$, i.e. computing $Tor$'s via the second alternative definition. Since there are no more place for non trivial differentials, $'E^2$ presents the limit of the spectral sequence. This shows that the second and the third alternative definitions for $Tor$'s give to the same result.

The second spectral sequence has $''E^2_{pq} = H^{\text{hor}}_p(H^{\text{ver}}_q(P_\bullet \otimes Q_\bullet))$ Now the arguments completely symmetric to the used above show that the columns $P_p \otimes Q_\bullet$ are acyclic everywhere besides the zero row, where we get the complex

$$\cdots \longrightarrow P_2 \otimes N \longrightarrow P_1 \otimes N \longrightarrow P_0 \otimes N \longrightarrow 0$$

which computes $Tor_\bullet(M,N)$ via the first definition. So, all three definitions lead to the same modules.

Exercice 6.17. Deduce from the above discussion that the canonical isomorphism $M \otimes N \simeq N \otimes M$, which interchanges factors in decomposable tensors, induces canonical isomorphisms $Tor_p(M,N) \simeq Tor_p(N,M)$ for all $p$.

In fact, there is even more freedom in a choice of resolutions computing $Tor(M,N)$.

[3] in a contrast with the previous lecture, we are using the bottom (i.e. «homological») indexation here and this reverses all the arrows: horizontal ones go from the right to the left and vertical go down

Claim 6.18. *Let* $\cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0$ *be any complex of modules whose non zero homology is exhausted by* $H_0(A_\bullet) = M$. *If* $Tor_p(A_\nu, N) = 0$ *for all* $p \geqslant 1$ *and all* $\nu$, *then* $Tor_p(M,N) = H_p(A_\bullet \otimes N)$, *i. e. the complex*[4] $A_\bullet$ *can be used instead of projective resolution for* $M$ *in order to compute* $Tor(M,N)$.

Proof. Let $P_{p,q}$ be a Cartan – Eilenberg projective resolution for the complex $A_\bullet$ described in Example 6.13. We are going to compute homologies of the total complex $Tot(P_{\bullet,\bullet} \otimes N)$, of the bicomplex obtained by tensoring $P_{p,q}$ by $N$. The first spectral sequence converging to these homologies has ${}'E^2_{pq} = H^{\text{ver}}_q(H^{\text{hor}}_p(P_{\bullet,\bullet} \otimes N))$. Since each row $P_{\bullet,q}$ is a projective resolution of a projective module $H^{\text{hor}}_0(P_{\bullet,q})$, the horizontal homologies $H^{\text{hor}}_p(P_{\bullet,q} \otimes N) = Tor_p(H^{\text{hor}}_0(P_{\bullet,q}), N)$ vanish for all $p > 0$. Hence $H^{\text{hor}}_p(P_{\bullet,q} \otimes N)$ are exhausted by $H^{\text{hor}}_0(P_{\bullet,q}) \otimes N$ staying in the zero column and forming a projective resolution for $H^0(A_\bullet) = M$ tensored by $N$. So, ${}'E^2_{pq} = {}' E^\infty_{pq}$ is situated in the zero column and has ${}'E^2_{0q} = Tor_q(M,N)$.

The second spectral sequence has ${}''E^2_{pq} = H^{\text{hor}}_p(H^{\text{ver}}_q(P_{\bullet,\bullet} \otimes N))$. But $p$-th column $P_{p,\bullet}$ gives a projective resolution for $A_p$. Hence $H^{\text{ver}}_q(P_{\bullet,\bullet} \otimes N) = Tor_q(A_p, N)$ vanishes for $q > 0$ by the assumption of the claim, i. e. the vertical homologies $H^{\text{ver}}_q(P_{\bullet,\bullet} \otimes N)$ are exhausted by the zero row, where we get a complex $A_p \otimes N$ whose homologies are ${}''E^2_{pq} = {}'' E^\infty_{pq}$. So, these homologies coincide with $Tor_\bullet(M,N)$. □

The most important property of higher torsions is given by the following

Claim 6.19. *For any module* $R$ *and any exact triple of modules* $0 \longrightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \longrightarrow 0$ *there is canonical long exact sequence of higher torsion modules*

$$\begin{aligned}\cdots \longrightarrow Tor_2(M,R) \longrightarrow Tor_2(N,R) \longrightarrow Tor_1(L,R) \longrightarrow Tor_1(M,R) \longrightarrow \\ \longrightarrow Tor_1(N,R) \longrightarrow L \otimes R \longrightarrow M \otimes R \longrightarrow N \otimes R \longrightarrow 0\end{aligned} \tag{38}$$

*which continues short exact sequence* (36).

Proof. Given a projective resolution $P_\bullet \twoheadrightarrow R$ for $R$, there is a commutative diagram diagram

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
0 \arrow[r] & L \otimes P_{i-1} \arrow[r, "\varphi\otimes\mathrm{Id}_{P_{i-1}}"] \arrow[d, "\mathrm{Id}_L\otimes\partial"'] & M \otimes P_{i-1} \arrow[r, "\psi\otimes\mathrm{Id}_{P_{i-1}}"] \arrow[d, "\mathrm{Id}_M\otimes\partial"'] & N \otimes P_{i-1} \arrow[r] \arrow[d, "\mathrm{Id}_N\otimes\partial"'] & 0 \\
0 \arrow[r] & L \otimes P_{i} \arrow[r, "\varphi\otimes\mathrm{Id}_{P_{i}}"] \arrow[d, "\mathrm{Id}_L\otimes\partial"'] & M \otimes P_{i} \arrow[r, "\psi\otimes\mathrm{Id}_{P_{i}}"] \arrow[d, "\mathrm{Id}_M\otimes\partial"'] & N \otimes P_{i} \arrow[r] \arrow[d, "\mathrm{Id}_N\otimes\partial"'] & 0 \\
0 \arrow[r] & L \otimes P_{i+1} \arrow[r, "\varphi\otimes\mathrm{Id}_{P_{i+1}}"] \arrow[d] & M \otimes P_{i+1} \arrow[r, "\psi\otimes\mathrm{Id}_{P_{i+1}}"] \arrow[d] & N \otimes P_{i+1} \arrow[r] \arrow[d] & 0 \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}$$

whose columns are obtained by tensoring the projective resolution $P_\bullet \twoheadrightarrow R$ by $L$, $M$, $N$ respectively (i. e. they compute $Tor_\bullet(M,R)$) and rows are exact triples obtained by tensoring the initial exact triple by projective modules $P_i$. So, we have an exact triple of complexes and its long snake exact sequence of homologies gives the required sequence of $Tor$'s. □

Example 6.20: flat modules. Claim 6.18 shows that most general resolution $A_\bullet \twoheadrightarrow M$ suitable to compute $Tor(M,N)$ *for any* module $N$ should consist of modules $A_\nu$ that have no higher torsion products with any $N$. A module $A$ is called *flat*, if $Tor_p(A,N) = 0\ \forall\, p \geqslant 1$ and all modules $N$. It follows from the previous claim that any exact triple of modules remains to be exact after the tensor multiplication by a flat module $A$. Since any long exact sequence can be glued from exact triples, exactness of any sequences is also preserved under tensoring by a flat module. On the other side, the previous claim implies that tensoring by $A$ preserves exactness as soon as $Tor_1(A,N) = 0$ for all $N$. So, $A$ is flat iff $Tor_1(A,N) = 0$ for all $N$.

WORKING EXAMPLE: RESOLUTIONS OF GRADED POLYNOMIAL IDEALS. A polynomial ideal

$$I \subset S = S^\bullet V^* \simeq k[x_0, x_1, \dots, x_n]$$

spanned by some homogeneous polynomials $f_1, f_2, \dots, f_m$ , $f_i \in S^{d_i}$, is called *homogeneous*. Its $d$-th homogeneous component $I^{(d)} = I \cap S^d$ coincides with an image of map

$$S^{d-d_1} \oplus S^{d-d_2} \oplus \cdots \oplus S^{d-d_m} \xrightarrow{(g_1,g_2,\dots,g_m)\mapsto\sum f_i g_i} S^d$$

[4] such a complex is called $N$-acyclic resolution of $N$

Expanding each polynomial $f = \sum h_i f_i \in I$ as a sum of its homogeneous components, we see immediately that $d$-th degree component has a form $\sum h_i^{(d-d_i)} f_i \in I^{(d)}$, where $h_i^{(d-d_i)}$ are homogeneous components of $h_i$. So, all homogeneous components of any polynomial from $I$ also belong to $I$ and $I = \bigoplus_d I^{(d)}$ as a vector space over $k$. This means that $I$ is *a graded module* over a graded ring $S$. More precisely, an $S$-module $M$ is called *graded*, if $M = \bigoplus_{d\in\mathbb{Z}} M_d$ as a vector space over $k$ and $S^r \cdot M^t \subset M^{r+t}\ \forall\, t, r$.

As soon as a graded $S$-module $M$ is spanned by homogeneous elements $f_i \in M^{d_i}$, we have (homogeneous, degree zero) epimorphism of graded $S$-modules:

$$\bigoplus_i S[-d_i] \xrightarrow{\partial} M$$

where $S[-d_i]$ is free $S$-module of rank 1 with basic element $e_i$ of degree $d_i$ (that is $S[-d_i]^d = S^{d-d_i}$) and $\partial$ takes $e_i \longmapsto f_i$. Generating system $f_1, f_2, \dots, f_m$ is called *minimal*, if no one out of $f_i$'s can be expressed through the other as $f_i = \sum_{j\neq i} h_j f_j$ with some $h_j \in S$. In more symmetric terms, this means that each linear relation $\sum g_\nu f_\nu = 0$, between the generators $f_\nu$, with homogeneous coefficients $g_\nu \in S^{d_\nu}$, has all $d_\nu > 0$. Clearly, any generating system can be reduced to some minimal one. Taking a minimal generating system for $\ker\partial$ and then iterating the procedure, we get *a minimal free resolution*

$$\cdots \xrightarrow{\partial_4} F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial} M \longrightarrow 0$$

where $F_p = \bigoplus_{i_p} S[-d_{i_p}]$ is a module of $p$-th order syzygies, whose generating elements are in 1–1 correspondence with some minimal generating system for $\ker\partial_{p-1}$, that is for relations between $(p-1)$-th order syzygies.

Example 6.21: minimal free resolution for the maximal ideal $(x_0, x_1, \dots, x_n) \subset S$ is given by the Koszul complex (27), which can be written as

$$0 \longrightarrow \Lambda^{n+1}V^* \otimes S[-n-1] \xrightarrow{\partial} \Lambda^n V^* \otimes S[-n] \xrightarrow{\partial} \cdots \xrightarrow{\partial} V^* \otimes S[-1] \xrightarrow{\partial} (x_0, x_1, \dots, x_n) \longrightarrow 0 \qquad (39)$$

where $\partial\left(x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p} \otimes f(x)\right) = \sum_{\nu=0}^{n} \frac{\partial}{\partial x_\nu}(x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}) \otimes x_\nu \cdot f(x)$.

Example 6.22: minimal free resolution for the ideal of Veronese curve. Recall that the Veronese curve $C_d$ of degree $d$ in $\mathbb{P}_d = \mathbb{P}(S^d U)$, where $\dim U = 2$, consists of all pure $d$-th powers $u^d \in S^d U$, where $u \in U$ (see Example 3.14). Fixing a base $\{x, y\}$ for $U$ and induced base $\left\{\binom{d}{m} \cdot x^m y^{d-m}\right\}$ for $S^d U$, we can describe $C_d$ as the set of degree $d$ polynomials $f = \sum a_\nu \cdot \binom{d}{\nu} x^\nu y^{d-\nu}$ such that all linear forms

$$\frac{\partial^{d-1} f}{\partial x^i \partial y^{d-1-i}} = d! \cdot (a_i y + a_{i+1} x)\,, \quad 0 \leqslant i \leqslant (d-1)\,,$$

are proportional, i.e. $\operatorname{rk}\begin{pmatrix} a_0 & a_1 & \dots & a_{d-1} \\ a_1 & a_2 & \dots & a_d \end{pmatrix} = 1$. In other words, the homogeneous ideal of the Veronese curve $I = I(C_d)$ is spanned by quadrics $q_{ij} = a_i a_j - a_{i+1} a_{j-1} \in k[a_0, a_1, \dots, a_d]$.

Claim 6.23. *Let $F_\bullet \twoheadrightarrow M$ be a minimal free resolution of a graded $S$-module $M$ and $k = S/(x_0, x_1, \dots, x_n)$ be 1-dimensional vector space on which all the variables $x_i$ act by the zero homomorphisms. Then $\operatorname{rk} F_p = \dim_k \operatorname{Tor}_p(M, k)$; in particular, the ranks of free modules $F_\nu$ do not depend on a choice of minimal resolution.*

Proof. Differentials of minimal free resolution $\partial_p : \bigoplus_i S[-d_i] \longrightarrow \bigoplus_j S[-d_j]$ send each basic vectors $e_i^p \in F_p$ to some homogeneous generator $\partial(e_i^p) = \sum h_{ji}^p e_j^{p-1}$ for $\ker\partial_{p-1}$. So, $\partial_p$ is presented by a matrix $\left(h_{ji}^p\right)$ of homogeneous polynomials whose degrees are strictly positive because of minimality of the resolution. Multiplication by such a matrix becomes the zero map modulo the ideal $(x_0, x_1, \dots, x_n)$. Hence, tensoring a minimal free resolution by $k$ over $S$ leads to a complex with zero differentials. Homologies of this complex coincide with its elements. But $S[-n] \otimes_S k \simeq k[-n]$ is a trivial module $k$ placed at $n$-degree. □

Corollary 6.24. (Hibert's theorem about syzygies). *Any graded ideal $I \subset k[x_0, x_1, \dots, x_n]$ admits a minimal free resolution of length $\leqslant (n+1)$, i.e. such a resolution*

$$0 \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow I \longrightarrow 0 \qquad (40)$$

*is finite and has $r \leqslant n+1$.*

Proof. Since the Koszul complex (39) gives a free resolution for $k$ of form

$$0 \longrightarrow K_{n+1} \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_0 \longrightarrow 0\,, \quad K_p = \Lambda^p V^* \otimes S[-p]\,,$$

we have $Tor_p(k,M) = 0$ for $p \geqslant n+2$ and any $S$-module $M$. But a free resolution (40) for $I$ leads to a free resolution

$$0 \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow S \longrightarrow S/I \longrightarrow 0$$

for the factor module $M = S/I$ (it has $F_0 = S$). This implies $\operatorname{rk} F_r = \dim Tor_r(M,k) = 0$ for $r \geqslant (n+2)$. □

<u>Functor *Ext*.</u> Let $P_\bullet \twoheadrightarrow M$ be a free resolution for a module $M$. Applying $\operatorname{Hom}(*,N)$ to the complex

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_0} P_0 \longrightarrow 0$$

we get a complex (with an *increasing* indexation):

$$0 \longrightarrow \operatorname{Hom}(P_0,N) \longrightarrow \operatorname{Hom}(P_1,N) \longrightarrow \operatorname{Hom}(P_2,N) \longrightarrow \cdots .$$

Its $p$-th cohomology $Ext^p(M,N) \stackrel{\text{def}}{=} H^p(\operatorname{Hom}(P_\bullet,N))$ is called *$p$-th order extension module* of $M$ by $N$.

As it was with $Tor$'s, the modules $Ext^p(M,N)$ do not depend on a choice of free (even projective) resolution for $M$, because different projective resolutions lead to homotopy equivalent complexes $\operatorname{Hom}(P_\bullet,N)$. We also have $Ext^p(P,N) = 0$ for all $p \geqslant 1$ and all $N$, if $P$ is projective.

Exercice 6.25. Show that for any module $R$ the functor $\operatorname{Hom}(*,R)$ takes each exact triple of modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

to a short exact sequence $0 \longrightarrow \operatorname{Hom}(N,R) \longrightarrow \operatorname{Hom}(M,R) \longrightarrow \operatorname{Hom}(L,R)$ whose rightmost arrow is in general non surjective.

Since the functor $\operatorname{Hom}(*,N)$ takes exact triple $0 \longrightarrow \operatorname{im}\partial_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ to a short exact sequence $0 \longrightarrow \operatorname{Hom}(M,N) \longrightarrow \operatorname{Hom}(P_0,N) \longrightarrow \operatorname{Hom}(\operatorname{im}\partial_1,N)$, the starting $Ext$-module is

$$Ext^0(M,N) = \ker\Big(\operatorname{Hom}(P_0,N) \longrightarrow \operatorname{Hom}(\operatorname{im}\partial_1,N)\Big) = \operatorname{Hom}(M,N)\,.$$

By this reason, higher $Ext$'s are called *right derived functors* for the functor $\operatorname{Hom}(*,N)$.

<u>Claim 6.26</u>. *For any module $P$, each exact triple of modules $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ produces a long exact sequence of extension modules*

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}(P,L) \longrightarrow \operatorname{Hom}(P,M) \longrightarrow \operatorname{Hom}(P,N) \longrightarrow Ext^1(P,L) \longrightarrow Ext^1(P,M) \longrightarrow \\ \longrightarrow Ext^1(P,N) \longrightarrow Ext^2(P,L) \longrightarrow Ext^2(P,M) \longrightarrow Ext^2(P,N) \longrightarrow \cdots\end{aligned} \tag{41}$$

*which continues the short sequence* (37). *In particular, $P$ is projective iff $Ext^1(P,L) = 0$ for any module $L$.*

Proof. A free resolution $P_\bullet \twoheadrightarrow P$ for a module $P$ gives an exact triple of complexes

$$0 \longrightarrow \operatorname{Hom}(P_\bullet,L) \longrightarrow \operatorname{Hom}(P_\bullet,M) \longrightarrow \operatorname{Hom}(P_\bullet,N) \longrightarrow 0 \tag{42}$$

whose horizontal triples are obtained by applying exact functors $\operatorname{Hom}(P_\nu,*)$ to the initial triple

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

and complexes staying in columns compute $Ext(P,L)$, $Ext(P,M)$, and $Ext(P,N)$. So, the required long exact sequence of $Ext$'s is a long snake sequence of cohomologies for the triple (42). □

<u>Claim 6.27</u>. *For any module $R$, each exact triple of modules $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ produces a long exact sequence of extension modules*

$$\begin{aligned}0 \longrightarrow \operatorname{Hom}(N,R) \longrightarrow \operatorname{Hom}(M,R) \longrightarrow \operatorname{Hom}(L,R) \longrightarrow Ext^1(N,R) \longrightarrow Ext^1(M,R) \longrightarrow \\ \longrightarrow Ext^1(L,R) \longrightarrow Ext^2(N,R) \longrightarrow Ext^2(M,R) \longrightarrow Ext^2(L,R) \longrightarrow \cdots\end{aligned} \tag{43}$$

*which continues the short sequence from Exercice 6.25.*

Proof. Let us apply $\operatorname{Hom}(*,R)$ to Cartan – Eilenberg resolution

$$0 \longrightarrow P_\bullet \longrightarrow P_\bullet \oplus Q_\bullet \longrightarrow Q_\bullet \longrightarrow 0$$

of the initial exact triple $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$. Since for each $i$ we have $\mathrm{Hom}(P_i \oplus Q_i, R) = \mathrm{Hom}(P_i, R) \oplus \mathrm{Hom}(Q_i, R)$, we get an exact triple of complexes

$$0 \longrightarrow \mathrm{Hom}(Q_\bullet, R) \longrightarrow \mathrm{Hom}(P_\bullet \oplus Q_\bullet, R) \longrightarrow \mathrm{Hom}(P_\bullet, R) \longrightarrow 0$$

computing $Ext(N,R)$, $Ext(M,R)$, $Ext(L,R)$. So, the required long exact sequence of $Ext$'s again follows from the long cohomological snake sequence. □

Claim 6.28. *Let a complex* $0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots$ *have only one non zero cohomology* $H^0(A^\bullet) = N$. *If* $Ext^p(M, A^\nu) = 0$ *for all* $p \geqslant 1$ *and all* $\nu$, *then* $Ext^p(M,N) = H^p(\mathrm{Hom}(M, A^\bullet))$ *can be computed as cohomologies of the complex obtained by applying the functor* $\mathrm{Hom}(M,*)$ *to* $A^\bullet$.

Proof. Take a projective resolution $P_\bullet \twoheadrightarrow R$ for $R$ and compute cohomologies of total complex associated with a (cohomological) bicomplex $K^{pq} = \mathrm{Hom}(P_p, A^q)$ situated in the first quadrant. The first spectral sequence converging to $H^{p+q}(Tot(K^{\bullet,\bullet}))$ has $'E^2_{pq} = H^q_{\mathrm{ver}}(H^p_{\mathrm{hor}}(K^{\bullet,\bullet}))$. Since each row $K^{\bullet,q} = \mathrm{Hom}(P_\bullet, A^q)$ computes $Ext^\bullet(M, A^q)$, the horizontal homologies $H^p_{\mathrm{hor}}(K^{\bullet,q}) = Ext^p(M, A^q)$ are situated only in the zero column and form a complex $\mathrm{Hom}(M, A^q)$ obtained by applying $\mathrm{Hom}(M,*)$ to $A^\bullet$. So, the total cohomologies $H^{p+q}(Tot(K^{\bullet,\bullet}))$ coincide with the cohomologies of this complex.

At the same time, the second spectral sequence, which has $''E^2_{pq} = H^p_{\mathrm{hor}}(H^q_{\mathrm{ver}}(K^{\bullet,\bullet}))$ consists of cohomologies of the complex $\mathrm{Hom}(P_\bullet, N)$ situated at the zero row, that is of modules $Ext^\bullet(M,N)$. Indeed, $p$-th column of $K^{\bullet,\bullet}$, i.e. $K^{p,\bullet} = \mathrm{Hom}(P_p, A^\bullet)$, is obtained by applying the functor $\mathrm{Hom}(P_p,*)$ to the complex $A^\bullet$. Since $P_p$ is projective, this functor preserves exactness of all exact sequences. Hence the vertical cohomologies $H^q_{\mathrm{ver}}(\mathrm{Hom}(P_p, A^\bullet))$ are exhausted by $H^0_{\mathrm{ver}} = \mathrm{Hom}(P_p, M)$ staying at the zero row and forming a complex obtained by applying $\mathrm{Hom}(*, M)$ to the projective resolution $P_\bullet$ of $R$. □

Injective modules and injective resolutions. The previous claim shows that a right resolution

$$0 \longrightarrow N \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

that allows to compute $Ext(M,N)$ as cohomologies of the complex $\mathrm{Hom}(M, I^\bullet)$ *for each* module $M$ should consist of modules $I$ satisfying $Ext^p(M,I) = 0$ for all $p \geqslant 1$ and all $M$. Such the modules are called *injective*. In fact, injectivity and projectivity are completely dual notions obtained from each other by reversing the arrows.

Exercice 6.29. Prove the following claim:

Claim 6.30. *The following conditions on a module $I$ are pairwise equivalent:*

1) *any $K$-module homomorphism $N \xrightarrow{\varphi} I$ can be extended onto any $M \supset N$, i.e. for any inclusion $N \overset{\alpha}{\hookrightarrow} M$ $\exists$ $M \xrightarrow{\psi} I$ such that $\psi\alpha = \varphi$;*
2) *functor $h_I : N \longmapsto \mathrm{Hom}(N, I)$ takes each exact triple $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ to exact triple $0 \longrightarrow \mathrm{Hom}(N,I) \longrightarrow \mathrm{Hom}(M,I) \longrightarrow \mathrm{Hom}(L,I) \longrightarrow 0$;*
3) *functor $h_I$ preserves exactness of any sequences;*
4) *$Ext^p(M, I) = 0$ for all $p \geqslant 1$ and all $M$;*
5) *$Ext^1(M, I) = 0$ for all $M$.* □

Example 6.31: $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules. Indeed, let $N \subset M$ be an extension of $\mathbb{Z}$-modules and $N \xrightarrow{f} \mathbb{Q}$ be any $\mathbb{Z}$-linear homomorphism. We are going to construct a $\mathbb{Z}$-linear homomorphism $M \xrightarrow{\psi} \mathbb{Q}$ extending $\varphi$, i.e. such that $\psi(n) = \varphi(n)\ \ \forall\, n \in N$ . The Zorn lemma (or any other equivalent transfinite reason) implies that without loss of generality we can assume that the factor $M/N$ is generated by one element, say by a class $m \pmod N$ of some $m \in M \setminus N$. So, it is enough to find $\psi(m) \in \mathbb{Q}$ such that $z\psi(m) = \varphi(zm)$ for each $z \in \mathbb{Z}$ such that $zm \in N$. But $I_m \overset{\mathrm{def}}{=} \{z \in \mathbb{Z} \mid zm \in N\}$ is an ideal in $\mathbb{Z}$. If it is zero, then there are no relations on $m$ and we can assign $\psi(m)$ arbitrarily. Otherwise, $I_m = (d)$ is a principal ideal spanned by some $d \neq 0$ and the required extension is give by the assignment $\psi(m) = \varphi(dm)/d$.

Exercice 6.32. Repeat the same arguments for $\mathbb{Q}/\mathbb{Z}$ instead of $\mathbb{Q}$.

The injective module $\mathbb{Q}/\mathbb{Z}$ has the following important optional feature: for any $\mathbb{Z}$-module $M$ and any element $m \in M$ there is a $\mathbb{Z}$-linear map $M \xrightarrow{\varphi} \mathbb{Q}/\mathbb{Z}$ such that $\varphi(m) \neq 0$. Indeed, since $\mathbb{Q}/ZZ$ is injective, it is enough to define this map only on a submodule $\mathbb{Z} \cdot m \subset M$ spanned by $m$. If $\mathbb{Z} \cdot m \simeq \mathbb{Z}$ is free, then we can take $\varphi(zm) = z/2 \bmod \mathbb{Z}$. If $\mathbb{Z} \cdot m \simeq \mathbb{Z}/d\mathbb{Z}$, then we have to take $\varphi(zm) = z/d \bmod \mathbb{Z}$.

In general case, injectivity of a $K$-module $I$ also means divisibility by all non invertible elements of $K$.

Lemma 6.33. *A module $I$ over an arbitrary commutative ring $K$ is injective iff for any ideal $J \subset K$ and any $K$-linear map $J \xrightarrow{\varphi} I$ there exists an element $m_{\varphi,J} \in I$ such that $\varphi(z) = z \cdot m_{\varphi,J}$ for all $z \in J$.*

Proof. If $I$ is injective, $\varphi$ can be extended to $K$-linear map $K \xrightarrow{\psi} I$ and we can take $m_{\varphi,J} = \psi(1)$. To prove the opposite implication, we have to show that any map $N \xrightarrow{f} I$ can be extended onto any comprehensive module $M \supset N$ spanned modulo $N$ by one element $m \in M \setminus N$. As in the Example 6.31 above, consider an ideal $J = \{z \in K \mid zm \in N\}$ and apply the assumption of lemma to $K$-linear map $J \longrightarrow I$ taking $z \longmapsto \varphi(zm)$. We get $m_J \in I$ such that $\varphi(zm) = z \cdot m_J$ for all $z \in J$. So, the assignment $\psi(m) = m_J$ gives the required extension. $\square$

Example 6.34: a module $I_K = \mathrm{Hom}_{\mathbb{Z}}(K, \mathbb{Q}/\mathbb{Z})$, of all $\mathbb{Z}$-linear maps $K \xrightarrow{\varkappa} \mathbb{Q}/\mathbb{Z}$, is naturally a $K$-module w. r. t. to the action $z\varkappa(x) \overset{\text{def}}{=} \varkappa(zx)$ for $z, x \in K$. We are going to prove that $I_K$ is injective and for any $K$–module $M$ and any $m \in M$ there is a $K$-linear map $M \xrightarrow{\varphi} I_k$ such that $\varphi(m) \neq 0$.

The first follows from the injectivity criterium stated in Lemma 6.33. Indeed, consider any ideal $J \subset K$ and any $K$-linear homomorphism $J \xrightarrow{\alpha} I$, which takes $z \in J$ to a $\mathbb{Z}$-linear map $K \xrightarrow{\alpha_z} \mathbb{Q}/\mathbb{Z}$ such that $\alpha_{xz}(y) = \alpha_z(xy) = \forall\, x, y \in K$. Note that a $\mathbb{Z}$-linear map $J \xrightarrow{\mathrm{ev}_1} \mathbb{Q}/\mathbb{Z}$ is given by prescription $z \longmapsto \alpha_z(1)$. Since $\mathbb{Q}/\mathbb{Z}$ is injective $\mathbb{Z}$-module, this map can be extended to $\mathbb{Z}$-linear map $K \xrightarrow{\varepsilon} \mathbb{Q}/\mathbb{Z}$ defined on the whole of $K$ and satisfying $\forall\ z \in J$, $x \in K$ the condition: $(z\varepsilon)(x) = \mathrm{ev}_1(xz) = \alpha_{xz}(1) = \alpha_z(x)$. So, $\alpha_z = z\varepsilon$ for all $z \in J$ as required.

To construct $M \xrightarrow{\varphi} I_K$ such that $\varphi(m) \neq 0$ for a given $m \in M$ it is enough to suppose that $M = K \cdot m = K/\mathrm{Ann}\,(m)$ is spanned by $m$. Pick up any $\mathbb{Z}$-linear homomorphism $M \xrightarrow{\psi} \mathbb{Q}/\mathbb{Z}$ with $\psi(m) \neq 0$ and define $\mathbb{Z}$-linear map $K \xrightarrow{\varphi_m} \mathbb{Q}/\mathbb{Z}$ as $x \longmapsto \psi(xm)$. Now a required map $M \xrightarrow{\varphi} I_K$ is obtained as $K$-linear extension of the assignment $m \longmapsto \varphi_m$, namely, for each $xm \in M$ we put $\varphi(xm) = x \cdot \varphi_m : y \longmapsto \psi(xym)$.

Corollary 6.35. *Each $K$-module $M$ is a submodule of an injective $K$-module and admits a right injective resolution:*

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots .$$

Proof. Let $I^0 = \prod\limits_{\varphi} I_K^{(\varphi)}$ be a direct sum of injective $K$-modules $I_K$ from Example 6.34 labelled by all $K$-linear homomorphisms $M \xrightarrow{\varphi} I_K$. Then the map $m \longmapsto \prod \varphi(m) \in \prod\limits_{\varphi} I_K^{(\varphi)}$ injectively sends $M$ into $I^0$. Then the construction can be repeated for $I^0/M$ instead of $M$. $\square$

Exercice 6.36. Given injective resolutions $M \hookrightarrow I^\bullet$, $N \hookrightarrow J^\bullet$, show that

a) any map $M \xrightarrow{\varphi} N$ can be lifted to a morphism of complexes $I^\bullet \xrightarrow{\psi} J^\bullet$ that induces $\varphi$ on the zero homology;
b) any two such liftings are homotopy equivalent;
c) an injective resolution for a given module is unique up to an isomorphism in the homotopy category.