Polarizations and contracions for (skew) polynomials[1]

(SKEW) SYMMETRIC TENSORS. A symmetric group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ permuting factors in the decomposable tensors: $\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \stackrel{\text{def}}{=} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \quad \forall\, \sigma \in \mathfrak{S}_n$. Subspaces

$$\mathsf{Skew}^n V = \{\, t \in V^{\otimes n} \mid \sigma(t) = \operatorname{sgn}(\sigma) \cdot t \quad \forall\, \sigma \in \mathfrak{S}_n \,\}$$
$$\mathsf{Sym}^n V = \{\, t \in V^{\otimes n} \mid \sigma(t) = t \quad \forall\, \sigma \in \mathfrak{S}_n \,\}$$

are called the spaces of *skew symmetric* and *symmetric* tensors.

Claim 3.1. *If* $\operatorname{char}(k) = 0$, *then, restricting the canonical factorization maps:* $V^{\otimes n} \xrightarrow{\pi_{skew}} \Lambda^n V$, $V^{\otimes n} \xrightarrow{\pi_{sym}} S^n V$, *onto the spaces of (skew) symmetric tensors, we get canonical isomorphisms:*

$$\mathsf{Skew}^n V \xrightarrow{\pi_{skew}} \Lambda^n V \qquad \textit{and} \qquad \mathsf{Sym}^n V \xrightarrow{\pi_{sym}} S^n V\,. \tag{12}$$

Proof. In the skew symmetric case, a base of $\mathsf{Skew}^n V$ is formed by the tensors

$$e_{\langle i_1, i_2, \ldots, i_n \rangle} \stackrel{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \cdot e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes \cdots \otimes e_{i_{\sigma(n)}}$$

(sum of all the tensor monomials sent to the basic Grassmannian monomial $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ by π_{skew}). So, $\pi_{\text{skew}}\left(e_{\langle i_1, i_2, \ldots, i_n \rangle}\right) = n!\, e_I$. In the symmetric case, let us write $e_{[m_1, m_2, \ldots, m_d]}$ for the sum of all tensor monomials containing m_1 factors e_1, m_2 factors e_2, ... , m_d factors e_d, where $\sum\limits_{\nu} m_\nu = n$. These monomials form one $\mathfrak{S}_n$-orbit, which consists of $\frac{n!}{m_1!\, m_2! \cdots m_d!}$ elements and collects all the decomposable tensors sent to $e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$ by π_{sym}. As above, the tensors $e_{[m_1, m_2, \ldots, m_d]}$ form a base for $\mathsf{Sym}^n V$ and $\pi_{\text{sym}}\left(e_{[i_1, i_2, \ldots, i_n]}\right) = \frac{n!}{m_1!\, m_2! \cdots m_d!} e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d}$. □

Exercice 3.2. Verify that the above sums $e_{\langle i_1, i_2, \ldots, i_n \rangle}$ and $e_{[m_1, m_2, \ldots, m_d]}$ really give the bases for $\mathsf{Sym}^n V$ (over any field of an arbitrary characteristic). Also note that if $\operatorname{char}(k) > 0$ divides n, then all these basic (skew) symmetric tensors are annihilated by factorization through (skew) symmetric relations.

Exercice 3.3. Verify that if $\operatorname{char}(k) = 0$, then $V^{\otimes n} = \mathcal{I}^{(n)}_{\text{skew}} \oplus \mathsf{Skew}^n V = \mathsf{Sym}^n V \oplus \mathcal{I}^{(n)}_{\text{sym}}$, where the projection $V^{\otimes n} \twoheadrightarrow \mathsf{Sym}^n V$ along $\mathcal{I}^{(n)}_{\text{sym}}$ is given by *the symmetrization map*

$$\operatorname{sym}_n \,:\, \tau \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma(\tau)$$

and the projection $V^{\otimes n} \twoheadrightarrow \mathsf{Skew}^n V$ along $\mathcal{I}^{(n)}_{\text{skew}}$ given by *the alternation map*

$$\operatorname{alt}_n \,:\, \tau \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \cdot \sigma(\tau)\,.$$

POLARIZATION OF (SKEW) POLYNOMIALS. The inverse maps to the isomorphisms (12) take

$$\begin{aligned} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} &\longmapsto \frac{1}{n!} \cdot e_{\langle i_1, i_2, \ldots, i_n \rangle} \\ e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d} &\longmapsto \frac{m_1!\, m_2! \cdots m_d!}{n!} \cdot e_{[m_1, m_2, \ldots, m_d]}\,. \end{aligned} \tag{13}$$

The both maps are called *complete polarizations* of (skew) polynomials and are denoted by $f \longmapsto \operatorname{pl}(f)$.

Example 3.4: (skew) polynomials and (skew) symmetric multilinear forms. Full polarization $\operatorname{pl}(f)$ of a (skew) homogeneous degree n polynomial f of one argument on V can be considered as a multilinear form of n arguments on V. It sends $(v_1, v_2, \ldots, v_n)$ to the full contraction $\widetilde{f}(v_1, v_2, \ldots, v_n) \stackrel{\text{def}}{=} \langle\, v_1 \otimes v_2 \otimes \cdots \otimes v_n \,,\, \operatorname{pl}(f) \,\rangle$ and has the same symmetry properties as f, because for all $\sigma \in \mathfrak{S}_n$, $t \in V^{\otimes n}$, $\xi \in V^{*\otimes n}$ we have $\left\langle\, \sigma(t) \,,\, \sigma(\xi) \,\right\rangle = \left\langle\, t \,,\, \xi \,\right\rangle$, which implies $\left\langle\, \sigma(t) \,,\, \xi \,\right\rangle = \left\langle\, t \,,\, \sigma^{-1}(\xi) \,\right\rangle$.

Exercice 3.5. Check that for a symmetric quadratic form $q(x) \in S^2 V^*$ we have

$$\widetilde{q}(x, y) = \frac{q(x+y) - q(x-y)}{4} = \frac{q(x+y) - q(x) - q(y)}{2} = \frac{1}{2} \sum_{\nu=1}^{\dim V} y_\nu \frac{\partial q}{\partial x_\nu}\,.$$

[1] WARNING!!! In this lecture we suppose that the ground field k is algebraically closed and has zero characteristic.

Since any multilinear form φ may be presented via full contraction

$$\varphi(v_1, v_2, \dots, v_n) = \left\langle\, v_1 \otimes v_2 \otimes \cdots \otimes v_n \,,\, \xi \,\right\rangle$$

with some $\xi \in V^{*\otimes n}$, the complete polarization identifies S^nV^* and Λ^nV^* with the spaces of all symmetric and skew symmetric multilinear forms $\underbrace{V \times V \times \cdots \times V}_{n} \longrightarrow k$, in n arguments on V.

Exercice 3.6. In symmetric case, show that a homogeneous degree n polynomial $f(x)$ (in one argument $x \in V$) coincides with the restriction of the corresponding symmetric multilinear form $\widetilde{f}(x_1, x_2, \dots, x_n)$ (in n arguments $x_\nu \in V$) onto the diagonal: $f(x) = \widetilde{f}(x,\, x,\, \dots\, ,\, x)$.

Example 3.7: duality on polynomials Using the complete polarization and the full contraction between $V^{\otimes n}$ and $V^{*\otimes n}$ we obtain (over a field of zero characteristic) a natural non degenerate pairing between $\Lambda^n(V)$ and $\Lambda^n\,(V^*)$ as well as between $S^n(V)$ and $S^n\,(V^*)$. Namely, for two (skew) polynomials f, in $e_i \in V$, and ξ, in $x_i \in V^*$, we put $\left\langle\, f \,,\, \xi \,\right\rangle \stackrel{\text{def}}{=} \left\langle\, \mathrm{pl}(f) \,,\, \mathrm{pl}(\xi) \,\right\rangle$.

Exercice 3.8. Let $\{e_1, e_2, \dots, e_n\} \subset V$ and $\{x_1, x_2, \dots, x_n\} \subset V^*$ be dual bases. Check that:

$$\left\langle\, e_I \,,\, x_J \,\right\rangle = \begin{cases} 1/n! & ,\ \text{for } I = J \\ 0 & ,\ \text{for } I \neq J \end{cases} \tag{14}$$

$$\left\langle e_1^{m_1} e_2^{m_2} \cdots e_d^{m_d} \,,\, x_1^{\ell_1} x_2^{\ell_2} \cdots x_d^{\ell_d} \right\rangle = \begin{cases} \frac{m_1!\, m_2! \cdots m_d!}{n!} & ,\ \text{if } m_\nu = \ell_\nu \ \forall \nu \\ 0 & ,\ \text{otherwise} \end{cases} \tag{15}$$

Partial derivatives (symmetric case) For any vector $v \in V$ and any polynomial $f \in S^nV^*$ the contraction $c_1^i(\mathrm{pl}(f) \otimes v) \in V^{*n-1}$ does not depend on the choice of index i in $\mathrm{pl}(f)$ to be contracted, because $\mathrm{pl}(f) \in \mathsf{Sym}^nV^*$ is symmetric. Its projection to $S^{n-1}V^*$ is called *a polar* of v w. r. t. f and is denoted by $\mathrm{pl}_v f$.

Exercice 3.9. Show that $\deg(f) \cdot \mathrm{pl}_v f = \partial_v f$, where ∂_v is the derivative in v-direction, which takes f to $\sum\limits_{i=1}^{\dim V} v_i \frac{\partial f}{\partial x_i}$ (here $v = \sum v_i e_i$, $\{e_i\}$ is a base for V and $\{x_i\}$ is a dual base for V^*).

Exercice 3.10. Show that $n! \cdot \widetilde{f}(v_1, v_2, \dots, v_n) = \partial_{v_1}\partial_{v_2} \cdots \partial_{v_n} f$ for any $f \in S^nV^*,\ \ v_1, v_2, \dots, v_n \ \in V$.

In particular, $\partial_v f$ does not depend on a base choice, is bilinear in v, f and satisfies the following symmetry property:

$$m_2!\, \partial_{v_1}^{m_1} f(v_2) = (m_1 + m_2)!\, \widetilde{f}(\underbrace{v_1, v_1, \dots\, , v_1}_{m_1}, \underbrace{v_2, v_2, \dots\, , v_2}_{m_2}) = m_1!\, \partial_{v_2}^{m_2} f(v_1)\,, \tag{16}$$

where $(m_1 + m_2) = n = \deg f$. In particular, the both, left and right sides, are bihomogeneous of bidegree (m_1, m_2) in (v_1, v_2).

Exercice 3.11. Show that *multiple polars*: $\mathrm{pl}_{\,v_1, v_2, \dots, v_m} f \stackrel{\text{def}}{=} \mathrm{pl}_{v_1}\mathrm{pl}_{v_2} \cdots \mathrm{pl}_{v_1} f$ are symmetric and multilinear in v_i and linear in f, that is come from a linear map $S^mV \otimes S^nV^* \longrightarrow S^{n-m}V^*$, which sends $v_1v_2 \cdots v_m \otimes f$ to $\mathrm{pl}_{\,v_1, v_2, \dots, v_m} f(w) = \widetilde{f}(v_1, v_2, \dots, v_m\,, w, w, \dots\,, w)$.

Example 3.12: Taylor formula. For $f \in S^nV^*$ the value $f(v + w)$ can be computed as the full contraction of f with $(v + w)^m = \sum\limits_m \binom{n}{m} v^m w^{n-m} \in S^nV$. This can be arranged as the *Taylor formula*:

$$f(v + w) = \sum_{\nu=0}^{n} \frac{1}{\nu!}\, \partial_w^\nu f(v)\ . \tag{17}$$

Example 3.13: *span*(f) for $f \in S^nV^*$, that is a minimal subspace $W \subset V^*$ such that $f \in S^nW$, coincides with *span*$(\mathrm{pl}(f))$ described in Claim 2.11 as an image of the contraction map $S^{n-1}V \otimes S^nV^* \longrightarrow V^*$. So, in terms of partial derivatives, *span*(f) is generated by the linear forms

$$\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_{n-1}}} f(x) \tag{18}$$

obtained from f by all possible $(n-1)$-fold differentiations.

Veronese variety in $\mathbb{P}_N = \mathbb{P}(S^nV^*)$, where $N = \binom{n}{d} - 1$ and $\dim V = d + 1$, consists of pure n-th powers of linear forms, that is, of $f(x) = \xi^n(x)$, where $\xi \in \mathbb{P}(V^*)$. It has a rational parameterization given by the Veronese map

$$\mathbb{P}(V^*) \xrightarrow{\ \xi \longmapsto \xi^n\ } \mathbb{P}(S^nV^*)\ .$$

Example 3.13 presents the Veronese variety by a system of quadratic equations. Namely, $f(x) = \xi^n(x)$ for some $\xi \in V^*$ iff *span*(f) is 1-dimensional subspace generated by ξ. So, f is pure n-th power iff any two linear

forms (18) are proportional to each other. If we arrange the coefficients of two linear forms as the rows of some $2 \times (d+1)$-matrix, then their proportionality means that all 2×2-minors of this matrix vanish.

Example 3.14: Veronese's curve. Let $\dim U = 2$, $\mathbb{P}_1 = \mathbb{P}(U^*)$, $\mathbb{P}_n = \mathbb{P}(S^n U^*)$, $\{t_0, t_1\}$ be a base of U^*, and $\binom{n}{i} t_0^i t_1^{n-i}$ for $0 \leqslant i \leqslant n$ be the corresponding base of $S^n V^*$. Then the Veronese embedding $\mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ sends a linear form $(\alpha_0 t_0 + \alpha_1 t_1)$ to

$$(\alpha_0 t_0 + \alpha_1 t_1)^n = \sum_i \alpha_0^i \alpha_1^{n-i} \cdot \binom{n}{i} t_0^i t_1^{n-i}$$

Its image is a rational curve $C_n \subset \mathbb{P}_n$ called *Veronese curve* or *rational normal curve* of degree n. If we use the coefficients $(a_0 \colon a_1 \colon \ \ldots \ \colon a_n)$ and $(\alpha_0 : \alpha_1)$ as homogeneous coordinates for a polynomial $\sum\limits_i a_i \cdot \binom{n}{i} t_0^i t_1^{n-i} \in S^n U^*$ and for a linear form $\alpha_0 t_0 + \alpha_1 t_1 \in \mathbb{P}(U^*)$, then C_n will be presented parametrically as $a_i = \alpha_0^i \alpha_1^{n-i}$.

On the other hand, for binary form $f(t) = \sum\limits_i a_i \cdot \binom{n}{i} t_0^i t_1^{n-i}$ the linear forms (18) are exhausted by

$$\frac{\partial^i}{\partial t_0^i} \frac{\partial^{n-1-i}}{\partial t_1^{i-1-n}} f(t) = a_i t_0 + a_{i+1} t_1 \quad , \quad \text{where} \quad 0 \leqslant i \leqslant (n-1) \ .$$

So, $f \in C_n$ iff $\det \begin{pmatrix} a_i & a_{i+1} \\ a_{j-1} & a_j \end{pmatrix} = a_i a_j - a_{i+1} a_{j-1} = 0$ for all $0 \leqslant i < j \leqslant n$, i. e. iff

$$\operatorname{rk} \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_1 & a_2 & \ldots & a_n \end{pmatrix} = 1 \ .$$

For $n = 2$ we get the Veronese quadric $a_0 a_2 = a_1^2$ in $\mathbb{P}_2$. For $n = 3$ we have a rational cubic curve, which is called *a twisted cubic*, given as an intersection of 3 quadrics $a_0 a_2 = a_1^2$, $a_1 a_3 = a_2^2$, $a_0 a_3 = a_1 a_2$.

Exercice 3.15. Draw the picture and discover that the first two quadrics are simple cones with vertices at $(0:0:0:1)$ and at $(1:0:0:0)$; the third quadric is Segre's one and is ruled by two line families; two cones have a common line element $a_1 = a_2 = 0$, which joins the vertices but does not lie on the Segre quadric; the line element $a_0 = a_1 = 0$ of the first cone and the one $a_2 = a_3 = 0$ of the second do lie on the Segre quadric in the same ruling family. So, any two of 3 quadrics are intersected along the twisted cubic and one more line, that is, the Veronese curve can not be given by use of 2 equations only!

Example 3.16: linear span of Veronese curve. Given an arbitrary hyperplane $\pi = \{a \in \mathbb{P}_n \mid \sum A_i a_i = 0\}$, the intersection $C_n \cap \pi$ consists of all $a = a(\alpha)$ that satisfy the equation $\sum\limits_i A_i \alpha_0^i \alpha_1^{n-i} = 0$ whose left side is non zero polynomial of degree n. So, a hyperplane section of Veronese's curve always consists[2] of n points counted with appropriate multiplicities (typically, of n *distinct* points). In particular, the linear span of any $(n+1)$ Veronese's curve points gives the whole of $\mathbb{P}_n$.

Corollary 3.17. [3] *To prove that some linear in f assertion holds for all polynomials f, it is enough to verify it only for all powers of all linear forms.* □

Exercice 3.18. Use Arongold principle to give another proof for Taylor formula (17).

Partial derivatives (skew-symmetric case). The contraction $c_1^i(\mathrm{pl}(f) \otimes v) \in V^{*n-1}$ between a vector $v \in V$ and a skew polynomial $f \in \Lambda^n V^*$ slightly depends on the choice of index i: it changes the sign when i is incremented or decremented by 1, because $\mathrm{pl}(f) \in \mathsf{Skew}^n V^*$ is skew symmetric. We choose $i = 1$ and write $\mathrm{pl}_v f$ for the projection of $c_1^1(\mathrm{pl}(f) \otimes v)$ into $\Lambda^{n-1} V^*$. Now consecutive polarizations anti commute: $\mathrm{pl}_v \mathrm{pl}_w f = -\mathrm{pl}_w \mathrm{pl}_v f$, i. e. the multilinear map

$$\underbrace{V \times V \times \cdots \times V}_{m} \times \Lambda^n V^* \xrightarrow{(v_1, v_2, \ldots, v_m, f) \mapsto \mathrm{pl}_{v_1} \mathrm{pl}_{v_2} \ldots \mathrm{pl}_{v_m} f} \Lambda^{n-m} V^*$$

comes from the linear map $\Lambda^m V \otimes \Lambda^n V^* \longrightarrow \Lambda^{n-m} V^*$.

Exercice 3.19. Show that $\deg f \cdot \mathrm{pl}_v f = \partial_v f$, where $\partial_v = \sum v_i \frac{\partial}{\partial x_i}$, as in Exercice 3.9, and verify the following properties of skew partial derivatives:

a) $\partial_v \partial_w = -\partial_w \partial_v$ (in particular, $\partial_v^2 = 0 \quad \forall\, v \in V$);

b) $\widetilde{f}(v_1, v_2, \ldots, v_n) = \frac{1}{n!} \partial_{v_1} \partial_{v_2} \cdots \partial_{v_n} f$ for any $f \in \Lambda^n V^*$, $v_1, v_2, \ldots, v_n \in V$.

c) $\partial_v (f_1 \wedge f_2) = (\partial_v f_1) \wedge f_2 - (-1)^{\deg f_1} f_1 \wedge (\partial_v f_2)$, in particular,

$$\partial_{e_{i_\nu}} x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} = (-1)^{\nu - 1} x_{i_1} \wedge \cdots \wedge x_{i_{\nu-1}} \wedge x_{i_{\nu+1}} \wedge \cdots \wedge x_{i_n} \ .$$

[2] recall that we suppose the ground field to be algebraically closed

[3] it is known as *Arongold principle*

Example 3.20: *span*(f) for grassmannian polynomial $f \in \Lambda^n V$ is linearly generated by partial derivatives

$$\partial_J f = \partial_{x_{j_{n-1}}} \partial_{x_{j_{n-2}}} \dots \partial_{x_{j_1}} f = n! \left\langle x_{j_1} \otimes x_{j_2} \otimes \dots \otimes x_{j_{n-1}} \,,\, \mathrm{pl}(f) \right\rangle$$

where $J = (j_1, j_2, \dots, j_{n-1}) \subset \{1,\, 2,\, \dots,\, d\}$ runs through all $\binom{d}{n-1}$ increasing ordered collections of $(n-1)$ indexes and $\{x_i\} \in V^*$ form a dual base to some base $\{e_i\} \in V$. If $f = \sum\limits_I \alpha_I\, e_I$ w. r. t. the last base, then

$$\partial_J f = n! \cdot \sum_{i \notin J} (-1)^{n-p(i,J)}\, \alpha_{J \sqcup \{i\}}\, e_i \tag{19}$$

where $p(i, J)$ is the number of place where i stays in the increasing permutation I of $J \sqcup \{i\}$ (because x_I is the only monomial in $\Lambda^n V^*$ whose complete contraction with e_I is non zero, see (14)).

Example 3.21: Plücker relations. A skew polynomial $f \in \Lambda^n V$ is called *completely decomposable*, if $f = v_1 \wedge v_2 \wedge \dots \wedge v_n$ is a product of n linear factors, or equivalently, if $\dim span(f) = n$ is minimal possible.

Exercice 3.22. Show that f is completely decomposable iff $f \wedge v = 0\ \forall\, v \in span(f)$.

So, like in Example 3.13 — Example 3.14, the set of completely decomposable polynomials $f \in \Lambda^n V$ is described by a system of quadratic equations $(\partial_J f) \wedge f = 0$. By (19), a basic monomial $e_K \in \Lambda^{n+1} V$ appears in

$$(\partial_J f) \wedge f = n! \cdot \left(\sum_{i \notin J} (-1)^{p(i,J)}\, \alpha_{J \sqcup \{i\}}\, e_i \right) \wedge \left(\sum_I a_I e_I \right)$$

as $n! \cdot \sum\limits_{\alpha=1}^{n+1} (-1)^{n-p(k_\alpha, J)} \cdot \alpha_{J \sqcup \{k_\alpha\}} \alpha_{K \setminus \{k_\alpha\}} \cdot e_{k_\alpha} \wedge e_{K \setminus \{k_\alpha\}}$, i. e. with the coefficient, which up to a constant factor equals

$$P_{JK}(f) \stackrel{\text{def}}{=} \sum_{i \in K \setminus (K \cap J)} (-1)^{p(i,J,K)}\, \alpha_{J \sqcup i}\, \alpha_{K \setminus i}\ , \tag{20}$$

where $p(i, J, K)$ is the sum of the place numbers where i stays in the increasing version of $J \sqcup \{i\}$ and in K.

Exercice 3.23. Show that this coefficient vanishes, if $J \subset K$

System of quadratic equations $P_{JK}(f) = 0$, which defines the variety of completely decomposable Grassmannian polynomials, is known as the *Plücker relations*. Note that these quadratic equations are not «independent», even not pairwise different (see Example 3.25 below).

Grassmannian variety $\mathrm{Gr}(m, d) = \mathrm{Gr}(m, V)$ is defined as the set of all m-dimensional vector subspaces in a given d-dimensional vector space[4] V and generalizes the projective space $\mathbb{P}_{d-1} = \mathrm{Gr}(1, d)$. In the projective geometric language, $\mathrm{Gr}(m, d)$ is the set of all $(m-1)$-dimensional projective subspaces in $\mathbb{P}_{d-1}$. There is a canonical bijection $\mathrm{Gr}(m, V) \simeq \mathrm{Gr}(d-m, V^*)$ induced by the usual duality sending $U \subset V$ to $\mathrm{Ann}\, U \subset V^*$ and wise versa.

From the other viewpoint, the Grassmannians can be considered as skew-symmetric counter-partners of the Veronese varieties. In particular, there is a *Plücker embedding*

$$\mathrm{Gr}(m, V) \stackrel{\mathfrak{u}}{\hookrightarrow} \mathbb{P}(\Lambda^m V)$$

that sends m-dimensional subspace $U \subset V$ to the 1-dimensional subspace $\Lambda^m U \subset \Lambda^m V$. If U is based by the vectors $\{u_1, u_2, \dots, u_m\} \subset U$, then $\mathfrak{u}(U) = u_1 \wedge u_2 \wedge \dots \wedge u_m$ up to proportionality, because taking an other base, say $v_i = \sum a_{ij} u_j$, we get

$$\mathfrak{u}(U) = v_1 \wedge v_2 \wedge \dots \wedge v_m = \det(a_{ij}) \cdot u_1 \wedge u_2 \wedge \dots \wedge u_m\ .$$

Exercice 3.24. Verify that the Plücker embedding really is injective.

Hint. Given $U_1 \neq U_2$, use a base of V coming from $U_1 \cap U_2$, $U_1/(U_1 \cap U_2)$, $U_2/(U_1 \cap U_2)$, $V/(U_1 + U_2)$ and construct some $\omega \in \Lambda^{d-m} V$ such that $\omega \wedge \Lambda^k U_1 = 0$ but $\omega \wedge \Lambda^k U_2 = \Lambda^d V$.

So, the Grassmannian $\mathrm{Gr}(m, V)$ can be identified with projective subvariety in $\mathbb{P}(\Lambda^m V)$, which consists of completely decomposable Grassmannian polynomials and is described by Plücker quadratic relations considered in Example 3.21.

Example 3.25: $\mathrm{Gr}(2, 4)$ and 2×2-minors of 2×4-matrices. Let $\dim V = 4$ and fix a base $\{e_1, e_2, e_3, e_4\} \subset V$. Then a subspace $U \subset V$ spanned by u_1, u_2 can be presented as 2×4-matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$ whose rows are the

[4] if the nature of V is important, we write $\mathrm{Gr}(m, V)$ instead of $\mathrm{Gr}(m, d)$

coordinates of u_1, u_2. This matrix is defined by U up to the left multiplication $A \mapsto C \cdot A$ by any $C \in GL_2(k)$ (this corresponds to a base change in U). The Plücker embedding sends A to

$$u_1 \wedge u_2 = \sum_{i<j} \det \begin{pmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{pmatrix} e_i \wedge e_j \ .$$

So, the homogeneous coordinates of $\mathfrak{u}(U) \in \mathbb{P}_5$ w. r. t. the base $\{e_{ij} = e_i \wedge e_j\}$ are six 2×2-minors

$$A_{ij} = \det \begin{pmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{pmatrix} \ ,$$

of A. In particular, we conclude that the left multiplications by $C \in GL_2$ doesn't effect on the ratios between 2×2-minors of A. An other claim: 6 numbers $\{A_{ij}\}$ give a collection of 2×2-minors for some 2×4 matrix iff they satisfy (maybe, after some renumbering) the Plücker relations. Potentially, we can write 4 ones, corresponding to all possible distributions of $4 = 3 + 1$ indexes $\{1, 2, 3, 4\}$ between K and J.

Exercice 3.26. Check that they all produce the same quadratic equation $A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0$.

Exercice 3.27. Deduce alternatively, from the skew-symmetric version of the Lagrange theorem (see Example 2.23), that for $\dim V = 4$ the Grassmannian quadratic form $\omega \in \Lambda^2 V$ is a product of two liner forms iff $\omega \wedge \omega = 0$.

HINT. In appropriate coordinates ω equals either $e_1 \wedge e_2$ or $e_1 \wedge e_2 + e_3 \wedge e_4$.

Exercice 3.28. Is there 2×4 - matrix with minors a) $\{2, 3, 4, 5, 6, 7\}$ b) $\{3, 4, 5, 6, 7, 8\}$?