T-STABILITIES AND T-STRUCTURES ON TRIANGULATED CATEGORIES

A. L. Gorodentsev, S. A. Kuleshov, A. N. Rudakov

20.01.2004

Abstract. We suggest a new concept of t-stability in a triangulated category. It generalizes the stability data introduced by T. Bridgeland in [6]. We study some links between t-stabilities and t-structures and obtain a complete classification of t-stabilities and bounded t-structures on the derived categories of coherent sheaves on a line and on an elliptic curve.

§ 1. Introduction

About ten years ago the authors tried to extend Giesecker's stability for torsion free sheaves and make it applicable to any objects in derived categories of coherent sheaves and to more generic linear triangulated categories. We hoped to obtain some functorial filtrations that would be useful to generalize and investigate various moduli problems. But we had not come to a satisfactory definition. The state of affairs was changed by recent papers [6], [7] where T. Bridgeland suggested the suitable generalization of the Harder–Narasimhan filtration and formulated a new concept of stability in a triangulated category based on a notion of "central charge" which could be considered as the complexified version of the Mumford's slope function $\mu = c_1/r$.

There are two important goals purposed by papers [6], [7]: on one hand, to give clear sense to physical stream of consciousness in the series of quite inspiring papers by M. R. Douglas (see, for example, [9] and references there in); on the other hand, to interpret autoequivalences of derived categories and the braid group action on exceptional bases as a kind of monodromy transformations that occur when the stability data move through an appropriate moduli space. In order to construct the suitable finite dimensional moduli space of stability data T. Bridgeland puts strong constraints on the central charge function, which is the main modular parameter in his theory. These constrains are not natural in the general context of the classical theory of coherent sheaves. For example, Bridgeland's stability data do not cover Giesecker's stability and some other important geometric filtrations on the derived category of sheaves.

S. Kuleshov is partially supported by INTAS OPEN-2000-269.
1991 *Mathematics Subject Classification.* ?????.

In this paper we combine Bridgeland's approach with our old ideas (partially published in [15]) and suggest a notion of stability in a triangulated category (t-stability for briefness) that is more generic than the stability data of T. Bridgeland. Roughly speaking, we exclude the central charge but focus on the functorial representation of objects by Postnikov towers (t-filtrations for briefness) analogous to "slicing" used in [6]. In particular, the Harder–Narasimhan filtrations for Giesecker type slopes, the Grothendieck filtration by pure torsion sheaves, and the orthogonal decompositions w. r. t. exceptional bases of triangulated categories can be treated uniformly as particular cases of canonical destabilizing t-filtrations in our sense.

The paper is organized as follows. In § 2 we remind the basic properties of stability data on an abelian category and several ways to set up these data. In § 3 we define t-stability in a triangulated category and consider some basic examples of t-stabilities in derived categories. In particular, the Giesecker stability and orthogonal decomposition of objects w. r. t. an exceptional base provide the examples of such t-stabilities on the derived category of coherent sheaves. In § 4 we study some general properties of the destabilizing Postnikov towers, which generalize the classical Harder–Narasimhan filtrations. In § 5 we introduce a partial order on the set of all t-stabilities on a given triangulated category and discuss refining and coarsening of t-stabilities. In § 6 we describe all the t-stabilities on the bounded derived category of coherent sheaves on projective line. As a by-product we obtain the complete list of bounded t-structures in $D^b(\mathcal{C}oh\,\mathbb{P}_1)$. In § 7 we get similar classification for a bounded derived category of coherent sheaves on elliptic curve.

§ 2. Stability data in abelian categories

The stability concept in the category of coherent sheaves was introduced by D. Mumford when he used the Geometric Invariant Theory (GIT) to construct the moduli spaces of vector bundles on an algebraic curve. Since then this notion has undergone several modifications and become adapted for producing compact families of coherent sheaves on various algebraic manifolds (see [10], [13], [14]).

In a general way, a construction of such a compact family is reduced to the construction of projective factor space for an action of some reductive group G on an appropriate vector space V (say, one can interpret each sheaf of the family as the zero cohomology sheaf of some complex of fixed rigid sheaves with varying differentials; then the required moduli space is the projectivization of the linear space of all differentials modulo the action of automorphisms of terms of the complex). After this reduction, the stable sheaves are those that correspond to closed G-orbits. Typically they form an open set which is compactified by the semistable sheaves whose orbits are not closed but their closures do not pass through zero.

Well known Hilbert's criterium for the (semi) stability of a given vector $v \in V$ readdresses the question to the analysis of weight decomposition of v w. r. t. the action of the maximal torus $T \subset G$. This weight decomposition usually can be determined in terms of the initial category of sheaves so the Hilbert criterium turns to a collection of inequalities on the discrete (i. e. topological) invariants of the sheaf $F(v)$, which corresponds to the G-orbit of v. Namely, one can associate with any torsion free sheaf F a vector[1] $\mu(F)$ (called *a slope*) such that F goes to a GIT-semistable G-orbit iff $\mu(E) \leqslant \mu(F)$ for any subsheaf $E \subset F$ (the vectors are

[1]possibly, with only one component, i. e. just a number

compared lexicographically). A semistable sheaf F is stable iff it is simple, i. e. $\mathrm{Hom}(F,F)=\mathbb{C}$.

By the construction, the components of the slope vector are some ratios c/r, of appropriate additive functions[2] on the Grothendieck group $K_0(\mathcal{C}oh)$ of the category of coherent sheaves. This structure of the slope components forces the following *seesaw property* of the slope: for any exact triple of torsion free sheaves

$$0\to E\to F\to G\to 0,$$

we have

$$\begin{aligned} \mu(E)<\mu(F) &\quad\Leftrightarrow\quad \mu(F)<\mu(G),\\ \mu(E)=\mu(F) &\quad\Leftrightarrow\quad \mu(F)=\mu(G),\\ \mu(E)>\mu(F) &\quad\Leftrightarrow\quad \mu(F)>\mu(G). \end{aligned} \tag{2.1}$$

The seesaw property implies the following two pure categorical corollaries:

(i) $\mathrm{Hom}(E,F)=0$, if E, F are semistable and $\mu(E)>\mu(F)$;

(ii) any torsion free sheaf X admits a canonical Harder–Narasimhan filtration

$$\begin{tikzcd} X=F^0X \arrow[d] & F^1X \arrow[l, hook'] \arrow[d] & F^2X \arrow[l, hook'] \arrow[d] & \cdots \arrow[l, hook'] & F^nX \arrow[l, hook'] \arrow[d] & F^{n+1}X=0, \arrow[l, hook'] \\ G_0 & G_1 & G_2 & & G_n & \end{tikzcd}$$

(vertical maps come from exact triples $0\to F^{i+1}X\to F^iX\to G_i\to 0$), where all the factors $G_i=F^iX/F^{i+1}X$ are semistable with increasing slopes: $\mu(G_i)<\mu(G_j)$ for all $i<j$.

This was taken as a definition of abstract stability data in an arbitrary abelian category in [15], where the first attempt to produce such a definition was done.

Definition 2.1. A stability structure on an abelian category $\mathcal{A}$ is a preorder on the objects of $\mathcal{A}$ such that the following *seesaw property* holds: for each exact sequence of nonzero objects $0\to A\to B\to C\to 0$ one of the following three alternative conditions is satisfied:

1) $A\prec B\Leftrightarrow A\prec C\Leftrightarrow B\prec C$;
2) $A\succ B\Leftrightarrow A\succ C\Leftrightarrow B\succ C$;
3) $A\asymp B\Leftrightarrow A\asymp C\Leftrightarrow B\asymp C$

A non zero object $A\in\mathcal{A}$ is called semistable, if $B\preceq A$ for any non zero subobject $B\subset A$.

The following two consequences of these definition were proven in [15].

Theorem 2.2. *Given a stability structure on an abelian category $\mathcal{A}$, then:*

1) $\mathrm{Hom}(B,A)=0$ *for any two semistable objects A, B satisfying $A\prec B$;*

2) *if $\mathcal{A}$ is weekly Artinian*[3] *and weekly Noetherian*[4]*, then for any object $X\in\mathcal{A}$ there exists a unique Harder–Narasimhan filtration*

$$\begin{tikzcd} X=F^0X \arrow[d] & F^1X \arrow[l, hook'] \arrow[d] & F^2X \arrow[l, hook'] \arrow[d] & \cdots \arrow[l, hook'] & F^nX \arrow[l, hook'] \arrow[d] & F^{n+1}X=0 \arrow[l, hook'] \\ G_0 & G_1 & G_2 & & G_n & \end{tikzcd}$$

[2]usually, the rank r appears in all the denominators

[3]these means finiteness of the chains $A_1\supset A_2\supset A_3\supset\dots$ that are increasing w. r. t. the stability preorder, i. e. $A_1\prec A_2\prec A_3\prec\dots$

[4]these means stabilization of the increasing chains $A_1\subset A_2\subset A_3\subset\dots$ that are monotonous (either increasing or decreasing) w. r. t. the stability preorder

whose factors $G_i = F^i/F^{i+1}$ are semistable and satisfy $G_i \prec G_j$ for all $i < j$.

Thus, each preorder with the seesaw property on an abelian category with some finiteness conditions provides us with a distinguished class of semistable objects and a canonical Harder–Narasimhan filtration on each object of the category. In particular, the classical notion of stability, which makes a sense initially for the torsion free sheaves only, can be extended in this way onto the whole of category of coherent sheaves: all we need to this aim is a proper ordering of objects. Traditional ordering tool is *a vector slope*, which is a slightly more multi-purposed device than the central charge introduced by T. Bridgeland in [6].

Definition 2.3. Let $\mathcal{A}$ be an abelian category and $K_0(\mathcal{A})$ be its Grothendieck group. A linearly independent system of additive functions $(x_0, \dots, x_r)\colon K_0(\mathcal{A}) \to \mathbb{Z}$ is called *positive*, if for any $A \in \mathcal{A}$ the following conditions hold:

$$\begin{aligned} &x_0(A) \geqslant 0, \\ &x_0(A) = 0 \Rightarrow x_1(A) \geqslant 0, \\ &x_0(A) = x_1(A) = 0 \Rightarrow x_2(A) \geqslant 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &x_0(A) = \dots = x_{r-1}(A) = 0 \Rightarrow x_r(A) > 0. \end{aligned} \tag{2.2}$$

A positive system is called *exhaustive*, if $x_0(A) = \dots = x_r(A) = 0 \Rightarrow A = 0$. Given a positive system of additive functions $(x_0, \dots, x_r)$ on $\mathcal{A}$ (not necessary exhaustive), *a slope* of an object $A \in \mathcal{A}$ w. r. t. to this system, is a vector

$$\gamma(A) = \Big(\underbrace{1, \dots, 1}_{s}, \nu\Big(-\frac{x_{s+1}(A)}{x_s(A)}\Big), \nu\Big(-\frac{x_{s+2}(A)}{x_s(A)}\Big), \dots, \nu\Big(-\frac{x_r(A)}{x_s(A)}\Big)\Big),$$

where $\nu\big(\frac{a}{b}\big) = \frac{1}{\pi} \operatorname{arcctg} \frac{a}{b}$, and $s = \min_i \{x_i(A) \neq 0\}$.

For example: $(\mathrm{rk}, \deg)$ is an exhaustive positive pair of additive functions on the category of coherent sheaves on a smooth algebraic curve; $(\mathrm{rk}, \deg, \chi(\mathcal{O}_S, \cdot\,))$ is a positive exhaustive triple of additive functions on the category of coherent sheaves on a smooth algebraic surface S as soon as Picard's number of S equals 1.

A slope orders the objects of $\mathcal{A}$ by the following prescription:

$$A \preceq B \Leftrightarrow \gamma(A) \leqslant \gamma(B)$$

(the vectors are compared lexicographically). This ordering satisfies the seesaw property, because an inequality between the slope components means the same inequality between ratios of additive functions x_i/x_0 and there is well known property of fractions with positive denominators: the mediant $(a+b)/(c+d)$ always lies between a/c and b/d.

So, any slope from Definition 2.3 produces an ordering satisfying Definition 2.1 and, as a consequence, provides the category with a class of semistable objects and a canonical Harder–Narasimhan filtration. Since the last set of data is what we are going to extend in a context of general triangulated categories, let us define it here more formally.

Definition 2.4. Let $\mathcal{A}$ be an abelian category and Φ be a linearly ordered set such that for any $\varphi \in \Phi$ an extension closed subcategory $\Pi_\varphi \subset \mathcal{A}$ is fixed. A pair $(\Phi, \{\Pi_\varphi\}_{\varphi\in\Phi})$ is called *stability data* on $\mathcal{A}$, if the following conditions hold:
(i) $\mathrm{Hom}_{\mathcal{A}}(\Pi_{\varphi'}, \Pi_{\varphi''}) = 0$ for all $\varphi' > \varphi''$;
(ii) each non zero object $X \in \mathcal{A}$ has a filtration

$$\begin{tikzcd} X = F^0X \arrow[d] & F^1X \arrow[l, hook'] \arrow[d] & F^2X \arrow[l, hook'] \arrow[d] & \cdots \arrow[l, hook'] & F^nX \arrow[l, hook'] \arrow[d] & F^{n+1}X = 0, \arrow[l, hook'] \\ G_0 & G_1 & G_2 & & G_n & \end{tikzcd}$$

where $G_i = F^iX/F^{i+1}X \in \Pi_{\varphi_i}$ and $\varphi_i < \varphi_j$ for all $i < j$, which is called the Harder–Narasimhan filtration.

We have seen that the existence of the Harder–Narasimhan filtration is the weakest property in the following hierarchy of data:

a slope built from ratios of additive functions
$\Downarrow$
an ordering satisfying the seesaw property
$\Downarrow$
set of semistable subcategories providing the Harder–Narasimhan filtration.

It would be very interesting to understand what freedom we have in recovering the higher flows of this hierarchy from the lower ones.

§ 3. Definition and basic examples of t-stability

In abelian category, a filtration on an object is a series of subobjects and their factors organized into a sequence of linked exact triples. In triangulated category, we have exact triangles instead of exact triples and an analog of a filtered object is a Postnikov tower

$$\begin{tikzcd}[column sep=small] & X_{\varphi_0} \arrow[dr, dashed] & & X_{\varphi_1} \arrow[dr, dashed] & & & & X_{\varphi_n} \arrow[dr, dashed] & \\ X = F^0X \arrow[ur, "q_0"] & & F^1X \arrow[ll, "p_1"] \arrow[ur, "q_1"] & & F^2X \arrow[ll, "p_2"] & \cdots \arrow[l] & F^nX \arrow[l] \arrow[ur, "q_n"] & & F^{n+1}X = 0, \arrow[ll, "p_{n+1}"] \end{tikzcd}$$

which consists of exact triangles

$$\begin{tikzcd} & X_{\varphi_i} \arrow[dr, dashed] & \\ F^iX \arrow[ur, "q_i"] & & F^{i+1}X. \arrow[ll, "p_{i+1}"] \end{tikzcd}$$

For shortnes, we call such a tower *a* t*-filtration* of an object X and refer to the objects F^iX and $X_i = X_{\varphi_i}$ as *the elements* and *the factors* of the t-filtration. If an object X admits a t-filtration with consequent factors $(X_0, X_1, \ldots, X_n)$, we write

$$X \rightsquigarrow (X_0, X_1, \ldots, X_n)$$

each time when only the factors but not the morphisms between them are important. Now, let us formulate our main definition.

Definition 3.1. Let $\mathcal{T}$ be a triangulated category and Φ be a linearly ordered set such that for any $\varphi \in \Phi$ a strictly full extension closed[5] non empty subcategory $\Pi_\varphi \subset \mathcal{T}$ is given. The data $(\Phi, \{\Pi_\varphi\}_{\varphi\in\Phi})$ are called t-*stability*, if

1) the grading shift functor $X \mapsto X[1]$ acts on Φ as a non decreasing automorphism, i. e. there is a bijection $\tau \in \operatorname{Aut}\Phi$ such that $\Pi_\varphi[1] = \Pi_{\tau(\varphi)}$ and $\tau(\varphi) \geqslant \varphi$ for all φ;

2) $\operatorname{Hom}^{\leqslant 0}(\Pi_\psi, \Pi_\varphi) = 0$ for all $\psi > \varphi$ from Φ;

3) each non zero object $X \in \mathcal{T}$ admits a finite t-filtration with strictly increasing semistable factors, i. e. there is a Postnikov tower

$$
\begin{tikzcd}
& X_{\varphi_0} \arrow[dr, dashed] & & X_{\varphi_1} \arrow[dr, dashed] & & & & X_{\varphi_n} \arrow[dr, dashed] & \\
X = F^0X \arrow[ur, "q_0"] & & F^1X \arrow[ll, "p_1"] \arrow[ur, "q_1"] & & F^2X \arrow[ll, "p_2"] & \cdots \arrow[l] & F^nX \arrow[l] \arrow[ur, "q_n"] & & F^{n+1}X = 0 \arrow[ll, "p_{n+1}"]
\end{tikzcd}
\tag{3.1}
$$

with non zero factors $X_{\varphi_i} \in \Pi_{\varphi_i}$ and strictly increasing $\varphi_i < \varphi_{i+1}$.

We call (3.1) *the Harder–Narasimhan filtration* of an object X. The factors X_{φ_i} are called *the semistable factors* of an object X. The categories Π_φ are called *the semistable subcategories* of the t-stability $(\Phi, \{\Pi_\varphi\}_{\varphi\in\Phi})$. We will often denote the whole set of the t-stability data by just one letter Φ.

3.1. t-filtrations induced by t-structures. The simplest example of a t-stability comes from a t-structure. Recall (see [8]) that a t-structure on a triangulated category $\mathcal{T}$ is a pair of strictly full subcategories $(D^{\leqslant 0}, D^{\geqslant 0})$ that satisfies the following properties:

1) $D^{\leqslant 0} \subset D^{\leqslant 0}[1]$ and $D^{\geqslant 0} \supset D^{\geqslant 0}[-1]$;

2) $\operatorname{Hom}^0(D^{\leqslant 0}, D^{\geqslant 0}[-1]) = 0$;

3) any $X \in \mathcal{T}$ can be fitted in an exact triangle

$$
\begin{tikzcd}
& X_{\geqslant 1} \arrow[dr, dashed] & \\
X \arrow[ur, "q"] & & X_{\leqslant 0} \arrow[ll, "p"]
\end{tikzcd}
$$

with $X_{\geqslant 1} \in D^{\geqslant 0}[-1]$ and $X_{\leqslant 0} \in D^{\leqslant 0}$.

If, besides these three properties, the following extra condition holds:

4) for each $X \in \mathcal{T}$ there are some $m, n \in \mathbb{Z}$ such that

$$X \in D^{\geqslant 0}[-m] \cap D^{\leqslant 0}[-n],$$

then the t-structure is called *bounded* (see [6]). As usually, we put

$$D^{\leqslant n} = D^{\leqslant 0}[-n] \qquad \text{and} \qquad D^{\geqslant n} = D^{\geqslant 0}[-n].$$

The full subcategory $\mathcal{A} = D^{\leqslant 0} \cap D^{\geqslant 0}$ is automatically abelian (in particular, extension closed). It is called *a core* of the t-structure.

[5]a subcategory $\mathcal{A} \subset \mathcal{T}$ is called *extension closed*, if for any $A, C \in \mathcal{A}$ the existence of an exact triangle $A \to B \to C \to A[1]$ forces $B \in \mathcal{A}$

Lemma 3.2. *Let $(D^{\leqslant 0}, D^{\geqslant 0})$ be a bounded* t*-structure on a triangulated category $\mathcal{T}$. Then the extension closed subcategories $\Pi_i = \mathcal{A}[i] = D^{\leqslant -i} \cap D^{\geqslant -i}$ provide the category $\mathcal{T}$ with a* t*-stability $(\mathbb{Z}, \{\Pi_i\}_{i\in\mathbb{Z}})$.*

Proof. The definition of t-structure implies $\mathrm{Hom}^{\leqslant 0}(D^{\leqslant -p}, D^{\geqslant -q}) = 0$ for $p < q$. Hence, the subcategories Π_i satisfy the required orthogonality conditions

$$\mathrm{Hom}^{\leqslant 0}(\Pi_i, \Pi_j) = 0 \quad \text{for} \quad i > j$$

and the grading shift acts on them by increasing shift of indeces. To construct the Harder–Narasimhan filtration of an arbitrary non zero object $X \subset \mathcal{T}$ note that there are two integers $n_+(X) \geqslant n_-(X)$ such that

$$\begin{aligned} &\mathrm{Hom}^0(X, D^{\geqslant -n_-(X)}) \neq 0, \qquad \mathrm{Hom}^{\leqslant 0}(X, D^{\geqslant -n}) = 0 \quad \text{for} \quad n < n_-(X), \\ &\mathrm{Hom}^0(D^{\leqslant -n_+(X)}, X) \neq 0, \qquad \mathrm{Hom}^0(D^{\leqslant -k}, X) = 0 \quad \text{for} \quad k > n_+(X). \end{aligned}$$

(they exist because the t-structure is bounded). We put $n_0 = n_-(X)$. By the definition of t-structure, there exists an exact triangle

$$\begin{tikzcd} & X_0 \arrow[dr, dashed] & \\ X \arrow[ur] & & Y_0 \arrow[ll] \end{tikzcd}$$

with $X_0 \in D^{\geqslant -n_0}$ and $Y_0 \in D^{\leqslant -(n_0+1)}$. It is easy to check that $X_0 \in D^{\geqslant -n_0} \cap D^{\leqslant -n_0} = \Pi_{n_0}$ is a non zero element and $\mathrm{Hom}^{\leqslant 0}(Y_0, X_0) = 0$. So, we can take $F^1 X = Y_0$ as the first term of the Harder–Narasimhan filtration we are constructing and then repeat the procedure inductively.

In fact, Lemma 3.2 shows how to extend arbitrary stability data on the abelian core $\mathcal{A}$ onto the whole of the triangulated category $\mathcal{T}$.

Proposition 3.3. *Let $(\Phi, \{\Pi_\varphi\}_{\varphi\in\Phi})$ be stability data (in the sense of Definition* 2.4*) on the core $\mathcal{A} = D^{\geqslant 0} \cap D^{\leqslant 0}$ of a bonded* t*-structure $(D^{\geqslant 0}, D^{\leqslant 0})$ on a triangulated category $\mathcal{T}$. We fix the lexicographical ordering on the set $\Psi = \mathbb{Z} \times \Phi$ and put $P_{(i,\varphi)} = \Pi_\varphi[i]$. Then the data $\big(\Psi, \{P_{(i,\varphi)}\}_{(i,\varphi)\in\Psi}\big)$ (with an evident action of the grading shift functor) give a* t*-stability on the triangulated category $\mathcal{T}$.*

Proof. Only the existence of finite Harder–Narasimhan filtration for any non zero object may be not obvious here. But such a filtration is constructed at once by means of the previous lemma and simple technical Proposition 4.3, which will be established in the next paragraph.

Corollary 3.4. *Any stability data (in the sense of Definition* 2.4*) $(\Phi, \{\Pi_\varphi\}_{\varphi\in\Phi})$ on an abelian category $\mathcal{A}$ canonically induce* t*-stability $\big(\mathbb{Z}\times\Phi, \{\Pi_\varphi[i]\}_{(i,\varphi)\in\mathbb{Z}\times\Phi}\big)$ on the bounded derived category $D^b(\mathcal{A})$.*

In particular, for any projective variety X there is a t-stability on[6] $D^b(\mathcal{C}oh\, X)$ that extends the Giesecker stability, which is well defined initially only for torsion

[6]we always write $D^b(\mathcal{C}oh\, X)$ for the bounded derived category of coherent sheaves on a projective variety X

free coherent sheaves by means of the slope $\gamma(F) = \chi_F/r(F)$, where $r(F)$ is the rank and χ_F is the Hilbert polynomial[7] of coherent sheaf F. To construct this extension, we have firstly to extend the Giesecker stability onto the whole of $\mathcal{C}oh\,X$ (as it was done in the paper [15]) and then to apply Corollary 3.4 for producing the Giesecker t-stability on $D^b(\mathcal{C}oh\,X)$. In § 7 this program will be realized in all details on a smooth curve algebraic curve X.

3.2. Exceptional t-stability. Let $\mathcal{T}$ be a linear triangulated category (this means that $\bigoplus_\nu \operatorname{Hom}^\nu(X,Y)$ are finite dimensional graded vector spaces for any pair of objects X, Y). Recall that an object $E \in \mathcal{T}$ is called *exceptional*, if $\operatorname{Hom}^\bullet(E,E)$ is the 1-dimensional algebra spanned by the identity map. An ordered collection of exceptional objects $(E_0, E_1, \dots, E_n)$ is called *exceptional* (or *semiorthogonal*), if $\operatorname{Hom}^\bullet(E_j, E_i) = 0$ as soon as $i < j$. Such a collection *spans* $\mathcal{T}$, if $\mathcal{T}$ is the smallest full triangulated subcategory that is closed under direct sums and contains all E_i. For example, the bounded derived category $D^b(\mathcal{C}oh\,\mathbb{P}_n)$ is generated by the exceptional collection of any $(n+1)$ consequent invertible sheaves

$$(\mathcal{O}(k), \mathcal{O}(k+1), \dots, \mathcal{O}(k+n)),$$

(we write $\mathcal{O}(m)$ for the line bundle of degree m on $\mathbb{P}_n$ and k is an arbitrary integer).

For any object X of a linear triangulated category $\mathcal{T}$ and any finite dimensional graded vector space $V^\bullet$ there is a well defined product $V^\bullet \otimes X \in \mathcal{T}$, which represents the functor $Y \longmapsto V^\bullet \otimes \operatorname{Hom}^\bullet(Y,X)$ and is isomorphic to the direct sum

$$V^\bullet \otimes X \simeq \bigoplus_\nu V^\nu \otimes X[-\nu].$$

If $\mathcal{T}$ is spanned by an exceptional collection, then the following theorem on orthogonal decompositions holds[8] (see [11]):

Theorem 3.5. *Let a linear triangulated category $\mathcal{T}$ be spanned by an exceptional collection $(E_0, E_1, \dots, E_n)$. Then each object $X \in \mathcal{T}$ is fitted into the following Postnikov tower (which is functorial in X):*

$$\begin{tikzcd}
 & V_0^\bullet \otimes E_0 \arrow[dr, dashed] & & V_1^\bullet \otimes E_1 \arrow[dr, dashed] & & & & V_n^\bullet \otimes E_n \arrow[dr, dashed] & \\
X \arrow[ur] & & F^1X \arrow[ll] \arrow[ur] & & F^2X \arrow[ll] & \cdots \arrow[l] & F^nX \arrow[l] \arrow[ur] & & 0. \arrow[ll]
\end{tikzcd}$$

Under the assumption of Theorem 3.5, write Δ for the naturally ordered set $\{0, 1, \dots, n\}$ and for each $i \in \Delta$ consider a full extension closed subcategory $\Upsilon_i \subset \mathcal{T}$ spanned by all $E_i[z]$, $z \in \mathbb{Z}$. Since $(E_0, E_1, \dots, E_n)$ is an exceptional collection, the subcategories Υ_i satisfy the orthogonality conditions $\operatorname{Hom}^{\leqslant 0}(\Upsilon_j, \Upsilon_i) = 0$ whenever $j > i$. The grading shift functor takes each Υ_i to itself. Theorem 3.5 says that each object $X \in \mathcal{T}$ admits a finite Harder–Narasimhan filtration with increasing factors laying in Υ_i's. So, we get

Proposition 3.6. *Let a linear triangulated category $\mathcal{T}$ be spanned by an exceptional collection $(E_0, E_1, \dots, E_n)$. Then $(\Delta, \{\Upsilon_i\}_{i\in\Delta})$ provide $\mathcal{T}$ with a* t*-stability.*

Note that the Harder–Narasimhan filtration is functorial in the case of the exceptional t-stability just described. The next paragraph will be devoted to general categorical properties of t-filtrations. In particular, we prove the uniqueness of the Harder–Narasimhan filtration.

[7] considered as lexicographically ordered collection of coefficients staying in accordance with the decreasing order of powers

[8] it generalizes the classical Beilinson theorem for projective spaces

§ 4. General properties of t-filtrations

Here we fix some t-stability $(\Phi, \{\Pi_\varphi\}_{\varphi\in\Phi})$ on a triangulated category $\mathcal{T}$.

Theorem 4.1. *A Harder–Narasimhan filtration of a given $X \in \mathcal{T}$ is unique up to a unique isomorphism of the Postnikov towers.*

Proof. Let an object $X \in \mathcal{T}$ have two Harder–Narasimhan filtrations:

(4.1)
$$\begin{tikzcd}
 & X_{\varphi_0} \arrow[dr, dashed] & & X_{\varphi_1} \arrow[dr, dashed] & & & & X_{\varphi_n} \arrow[dr, dashed] & \\
X = F^0X \arrow[ur, "q_0"] & & F^1X \arrow[ll, "p_1"] \arrow[ur, "q_1"] & & F^2X \arrow[ll, "p_2"] & \cdots \arrow[l] & F^nX \arrow[l] \arrow[ur, "q_n"] & & F^{n+1}X = 0 \arrow[ll, "p_{n+1}"]
\end{tikzcd}$$

and

(4.2)
$$\begin{tikzcd}
 & X_{\psi_0} \arrow[dr, dashed] & & X_{\psi_1} \arrow[dr, dashed] & & & & X_{\psi_m} \arrow[dr, dashed] & \\
X = Q^0X \arrow[ur, "q'_0"] & & Q^1X \arrow[ll, "p'_1"] \arrow[ur, "q'_1"] & & Q^2X \arrow[ll, "p'_2"] & \cdots \arrow[l] & Q^mX \arrow[l] \arrow[ur, "q'_m"] & & Q^{m+1}X = 0 \arrow[ll, "p'_{m+1}"]
\end{tikzcd}$$

We have to show that $n = m$, $\varphi_i = \psi_i$ for all i, and the identity map $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{T}}(X, X)$ admits a unique extension to an isomorphism of t-filtrations

$$\begin{tikzcd}
 & X_{\varphi_0} \arrow[dr, dashed] \arrow[dd, "\cong"] & & X_{\varphi_1} \arrow[dr, dashed] \arrow[dd, "\cong"] & & & & X_{\varphi_n} \arrow[dr, dashed] \arrow[dd, "\cong"] & \\
X \arrow[ur, "q_0"] \arrow[dd, "\mathrm{Id}_X"', "\cong"] & & F^1X \arrow[ll] \arrow[ur, "q_1"] \arrow[dd, "\cong"] & & F^2X \arrow[ll] \arrow[dd, "\cong"] & \cdots \arrow[l] & F^nX \arrow[l] \arrow[ur, "q_n"] \arrow[dd, "\cong"] & & F^{n+1}X \arrow[ll] \arrow[dd, "\cong"] \\
 & X_{\psi_0} \arrow[dr, dashed] & & X_{\psi_1} \arrow[dr, dashed] & & & & X_{\psi_n} \arrow[dr, dashed] & \\
X \arrow[ur, "q'_0"] & & Q^1X \arrow[ll] \arrow[ur, "q'_1"] & & Q^2X \arrow[ll] & \cdots \arrow[l] & Q^nX \arrow[l] \arrow[ur, "q'_n"] & & Q^{n+1}X \arrow[ll]
\end{tikzcd}$$

To this aim we need the following general property of t-filtrations.

Proposition 4.2. *For any* t*-filtration*

$$\begin{tikzcd}
 & X_{\varphi_0} \arrow[dr, dashed] & & X_{\varphi_1} \arrow[dr, dashed] & & & & X_{\varphi_n} \arrow[dr, dashed] & \\
X = F^0X \arrow[ur] & & F^1X \arrow[ll] \arrow[ur] & & F^2X \arrow[ll] & \cdots \arrow[l] & F^nX \arrow[l] \arrow[ur] & & F^{n+1}X = 0. \arrow[ll]
\end{tikzcd}$$

of an object X the following properties hold:

1) $\mathrm{Hom}^{\leqslant 0}(X, \Pi_\varphi) = 0$, *if $\varphi < \varphi_0$;*
2) $\mathrm{Hom}^{\leqslant 0}(F^iX, \Pi_\varphi) = 0$, *if $\varphi \leqslant \varphi_i$;*
3) $\mathrm{Hom}^{\leqslant 0}(\Pi_\psi, X) = 0$, *if $\psi > \varphi_n$;*

4) *if an object* Y *has a Harder–Narasimhan filtration*

$$\begin{tikzcd}[column sep=small]
 & Y_{\psi_0} \arrow[dr, dashed] & & Y_{\psi_1} \arrow[dr, dashed] & & & & Y_{\psi_m} \arrow[dr, dashed] & \\
Y = F^0 Y \arrow[ur] & & F^1 Y \arrow[ll] \arrow[ur] & & F^2 Y \arrow[ll] & \cdots \arrow[l] & F^m Y \arrow[l] \arrow[ur] & & F^{m+1} Y = 0 \arrow[ll]
\end{tikzcd}$$

with $\varphi_n < \psi_0$, *then* $\mathrm{Hom}^{\leqslant 0}(Y, X) = 0$.

Proof. Just apply the functors $\mathrm{Hom}(\,\cdot\,, \Pi_\varphi)$, $\mathrm{Hom}(\Pi_\psi, \,\cdot\,)$, and $\mathrm{Hom}(Y, \,\cdot\,)$ to consequent exact triangles

$$\begin{tikzcd}
 & X_{\varphi_i} \arrow[dr, dashed] & \\
F^i X \arrow[ur] & & F^{i+1} X \arrow[ll]
\end{tikzcd}$$

and use the orthogonality conditions between $F^i X \in \Pi_{\varphi_i}$.

Now, let us return to the proof of Theorem 4.1. It is the standard consequence from the vanishing condition $\mathrm{Hom}^{\leqslant 0}(F^1 X, X_{\varphi_0}) = 0$ that the first exact triangle of the tower (4.1)

$$\begin{tikzcd}
 & X_{\varphi_0} \arrow[dr, dashed] & \\
X \arrow[ur, "q_0"] & & F^1 X, \arrow[ll]
\end{tikzcd}$$

(here $X_{\varphi_0} \in \Pi_{\varphi_0}$) is defined by its middle term X uniquely up to a unique isomorphism of triangles[9].

Taking an arbitrary object $Y \in \Pi_{\varphi_0}$ and applying the functor $\mathrm{Hom}(\,\cdot\,, Y)$ to the first exact triangle in (4.1), we get a long exact sequence

$$\mathrm{Hom}^{-1}(F^1 X, Y) \to \mathrm{Hom}^0(X_{\varphi_0}, Y) \xrightarrow{h_Y} \mathrm{Hom}^0(X, Y) \to \mathrm{Hom}^0(F^1 X, Y)\,,$$

whose leftmost and rightmost terms vanish by Proposition 4.2, (2). Hence,

$$\mathrm{Hom}^0(X_{\varphi_0}, Y) \cong \mathrm{Hom}^0(X, Y) \quad \forall\, Y \in \Pi_{\varphi_0}. \tag{4.3}$$

This means that an object X_{φ_0} represents the functor

$$\mathrm{Hom}^0(X, \,\cdot\,) \,:\, \Pi_{\varphi_0} \to \mathcal{V}ect.$$

Let us substitute X_{φ_0} instead of Y in (4.3) then q_0 coincides with the morphism

$$h_{X_{\varphi_0}}(\mathrm{Id}_{X_{\varphi_0}}) \in \mathrm{Hom}^0(X, X_{\varphi_0}).$$

Since $q_0 \neq 0$, we have $\mathrm{Hom}^0(X, \Pi_{\varphi_0}) \neq 0$. The same arguments applied to the first exact triangle of tower (4.2) show that $\mathrm{Hom}^0(X, X_{\psi_0}) \neq 0$ as well.

[9] compare this, for example, with [8, Lemma IV.4.5]

So, $\varphi_0 = \psi_0$ by Proposition 4.2, (1) applied to X, Π_{φ_0}, Π_{ψ_0}. Moreover, since the representing object is unique up to a unique isomorphism, the identity map $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{T}}(X, X)$ admits a unique extension to an isomorphism of the triangles

$$\begin{tikzcd}
X_{\varphi_0} \arrow[rrr, "\sim", "\exists!"'] \arrow[dd, dashed] & & & X_{\psi_0} \arrow[dd, dashed] \\
& X \arrow[ul, "q_0"'] \arrow[r, "\sim", "\mathrm{Id}_X"'] & X \arrow[ur, "q_0'"] & \\
F^1X \arrow[ur] \arrow[rrr, "\sim", "\exists!"'] & & & Q^1X \arrow[ul]
\end{tikzcd} .$$

Now, to finish the proof it is enough to repeat all the arguments consequently to the remaining triangles of tower (4.1).

For further purposes we need a simple technical result about arbitrary Postnikov towers (not related with t-stabilities). It allows to glue t-filtrations together and was used already in the proof of Proposition 3.3.

Proposition 4.3. 1) *Given a* t*-filtration* $X \rightsquigarrow (Y_0, Y_1, \ldots, Y_n)$ *of an object* X *and some* t*-filtrations* $Y_\nu \rightsquigarrow (X_{(\nu,0)}, \ldots, X_{(\nu,k_\nu)})$ *of all* Y_ν*'s, then* X *admits* t*-filtration*

$$X \rightsquigarrow \big(X_{(0,0)}, \ldots, X_{(0,k_0)}, X_{(1,0)}, \ldots, X_{(1,k_1)}, \ldots, X_{(n,0)}, \ldots, X_{(n,k_n)}\big).$$

2) *Vise versa, given* t*-filtration of the form*

$$X \rightsquigarrow \big(X_{(0,0)}, \ldots, X_{(0,k_0)}, X_{(1,0)}, \ldots, X_{(1,k_1)}, \ldots, X_{(n,0)}, \ldots, X_{(n,k_n)}\big),$$

then there is t*-filtration* $X \rightsquigarrow (Y_0, Y_1, \ldots, Y_n)$ *whose factors have* t*-filtrations of the form* $Y_s \rightsquigarrow (X_{(s,0)}, \ldots, X_{(s,k_s)})$ *for all* s*.*

3) *Given two* t*-filtered objects* $X \rightsquigarrow (X_0, X_1, \ldots, X_n)$, $\quad Y \rightsquigarrow (Y_0, Y_1, \ldots, Y_m)$, *then for any shuffle permutation of their factors*[10] $(Z_0, Z_1, \ldots, Z_{n+m+1})$ *there exists a* t*-filtration* $\;X \oplus Y \rightsquigarrow (Z_0, Z_1, \ldots, Z_{n+m+1})$*.*

Proof. We prove the assertion 2) at first. Let the t-filtration of X consist of consequent exact triangles

$$\begin{tikzcd}
& X_{(i,j)} \arrow[dr, dashed] & \\
F^{(i,j)}X \arrow[ur] & & F^{(i,j+1)}X, \arrow[ll, "p_{(i,j+1)}"]
\end{tikzcd}$$

with $i = 0, \ldots, n$, $j = 0, \ldots, k_i - 1$ and triangles

$$\begin{tikzcd}
& X_{(i,k_i)} \arrow[dr, dashed] & \\
F^{(i,k_i)}X \arrow[ur] & & F^{(i+1,0)}X, \arrow[ll, "p_{(i+1,0)}"]
\end{tikzcd}$$

[10] i. e. a linear ordering on $\{Z_s\}_{s=0,\ldots,n+m+1} = \{X_i\}_{i=0,\ldots,n} \sqcup \{Y_j\}_{j=0,\ldots,m}$ that preserves the initial orders of $\{X_i\}_{i=0,\ldots,n}$ and $\{Y_j\}_{j=0,\ldots,m}$

$i = 0, \dots, n$ (here $F^{(0,0)}X = X$ and $F^{(n+1,0)}X = 0$). Let us write

$$p_i : F^{(i+1,0)}X \to F^{(i,0)}X$$

for the composition $p_{(i+1,0)}p_{(i,k_i)} \dots p_{(i,1)}$ and Y_i for a cone of p_i. Thus, we get a new t-filtration $X \rightsquigarrow (Y_0, Y_1, \dots, Y_n)$, which consists of exact triangles

$$\begin{tikzcd} & Y_i \arrow[dr, dashed] & \\ F^{(i,0)}X \arrow[ur] & & F^{(i+1,0)}X \arrow[ll, "p_i"] \end{tikzcd},$$

and it remains to construct some t-filtration $Y_i \rightsquigarrow (X_{(i,0)}, \dots, X_{(i,k_i)})$ for each $i = 0, \dots, n$. Since there is t-filtration

$$F^{(i,0)}X \rightsquigarrow \big(X_{(i,0)}, \dots, X_{(i,k_i)}, X_{(i+1,0)}, \dots, X_{(n,k_n)}\big),$$

the following lemma gives us all we need.

Lemma 4.4. *Given a* t*-filtration of an object* Z

$$\begin{tikzcd} & Z_0 \arrow[dr, dashed] & & Z_1 \arrow[dr, dashed] & & & & Z_m \arrow[dr, dashed] & \\ Z = F^0Z \arrow[ur] & & F^1Z \arrow[ll, "h_1"] \arrow[ur] & & F^2Z \arrow[ll, "h_2"] & \cdots \arrow[l] & F^mZ \arrow[l] \arrow[ur] & & F^{m+1}Z = 0 \arrow[ll, "h_{m+1}"] \end{tikzcd},$$

for $s = 1, 2, , \dots, (m+1)$ *let us write* $c_s : F^sZ \to Z$ *for the composition* $h_s h_{s-1} \dots h_1$ *and form an exact triangle*

$$\begin{tikzcd} & Q_s \arrow[dr, dashed] & \\ Z \arrow[ur] & & F^sZ \arrow[ll, "c_s"] \end{tikzcd}.$$

Then Q_s *has a* t*-filtration* $Q_s \rightsquigarrow (Z_0, Z_1, \dots, Z_{s-1})$.

Proof. Induction over s. The base case $s = 1$ is evident. In general case we have two exact triangles represented by the horizontal and vertical lines in the following diagram

$$\begin{tikzcd} & Z_s & \\ Q_s[-1] \arrow[r] & F^sZ \arrow[u] \arrow[r, "c_s"] & Z \\ & F^{s+1}Z \arrow[u, "h_{s+1}"] & \end{tikzcd}.$$

By the octahedron axiom, they are fitted into diagram

$$\begin{tikzcd} & Z_s \arrow[r] & Q_{s+1} \\ Q_s[-1] \arrow[ur] \arrow[r] & F^sZ \arrow[u] \arrow[r, "c_s"] & Z \arrow[u] \\ & F^{s+1}Z \arrow[u, "h_{s+1}"] \arrow[ur, "c_{s+1}"'] & \end{tikzcd},$$

which contains two additional exact triangles

$$
\begin{tikzcd}
& Q_{s+1} \arrow[dr, dashed] & \\
Z \arrow[ur] & & F^{s+1}Z, \arrow[ll, "c_{s+1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& Q_s \arrow[dr, dashed] & \\
Q_{s+1} \arrow[ur] & & Z_s. \arrow[ll]
\end{tikzcd}
$$

By the inductive assumption, there is a t-filtration $Q_s \rightsquigarrow (Z_0, \dots, Z_{s-1})$, which starts with the exact triangle

$$
\begin{tikzcd}
& Z_0 \arrow[dr, dashed] & \\
Q_s \arrow[ur] & & F^1Q_s \arrow[ll]
\end{tikzcd}
$$

and finishes by t-filtration $F^1Q_s \rightsquigarrow (Z_1, \dots, Z_{s-1})$. To construct a required t-filtration $Q_{s+1} \rightsquigarrow (Z_0, Z_1, \dots, Z_{s-1}, Z_s)$, we apply the octahedron axiom to exact triangles

$$
\begin{tikzcd}
& Z_0 & \\
Q_{s+1} \arrow[r] & Q_s \arrow[u] \arrow[r] & Z_s[1] \\
& F^1Q_s \arrow[u] &
\end{tikzcd}
$$

and get the diagram

$$
\begin{tikzcd}
& Z_0 \arrow[r] & F^1Q_{s+1}[1] \\
Q_{s+1} \arrow[ur] \arrow[r] & Q_s \arrow[u] \arrow[r] & Z_s[1] \arrow[u] \\
& F^1Q_s \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

which contains the first factor F^1Q_{s+1} of required t-filtration as well as two exact triangles

$$
\begin{tikzcd}
& F^1Q_s \arrow[dr, dashed] & \\
F^1Q_{s+1} \arrow[ur] & & Z_s, \arrow[ll]
\end{tikzcd}
\qquad
\begin{tikzcd}
& Z_0 \arrow[dr, dashed] & \\
Q_{s+1} \arrow[ur] & & F^1Q_{s+1}. \arrow[ll]
\end{tikzcd}
$$

Combining the first of them with the inductive assumption, we obtain a t-filtration $F^1Q_{s+1} \rightsquigarrow (Z_1, \dots, Z_{s-1}, Z_s)$. Hence, the second triangle is fitted into t-filtration

$$
\begin{tikzcd}[column sep=small]
& Z_0 \arrow[dr, dashed] & & Z_1 \arrow[dr, dashed] & & & & Z_s \arrow[dr, dashed] & \\
Q_{s+1} = F^0Q_{s+1} \arrow[ur] & & F^1Q_{s+1} \arrow[ll] \arrow[ur] & & F^2Q_{s+1} \arrow[ll] & \cdots \arrow[l] & F^sQ_{s+1} \arrow[l] \arrow[ur] & & F^{s+1}Q_{s+1} = 0. \arrow[ll]
\end{tikzcd}
$$

This proves the lemma together with the assertion (2) of Proposition 4.3.

The proof of the assertion (1) is similar.

To proof the assertion (3), consider t-filtrations

$$\begin{tikzcd}[column sep=small]
& X_0 \arrow[dr, dashed] & & X_1 \arrow[dr, dashed] & & & & X_n \arrow[dr, dashed] & \\
X = F^0X \arrow[ur, "q_0"] & & F^1X \arrow[ll] \arrow[ur, "q_1"] & & F^2X \arrow[ll] & \cdots \arrow[l] & F^nX \arrow[l] \arrow[ur, "q_n"] & & F^{n+1}X = 0, \arrow[ll]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& Y_0 \arrow[dr, dashed] & & Y_1 \arrow[dr, dashed] & & & & Y_m \arrow[dr, dashed] & \\
Y = F^0Y \arrow[ur, "p_0"] & & F^1Y \arrow[ll] \arrow[ur, "p_1"] & & F^2Y \arrow[ll] & \cdots \arrow[l] & F^mY \arrow[l] \arrow[ur, "p_m"] & & F^{m+1}Y = 0. \arrow[ll]
\end{tikzcd}$$

and assume that

$$(Z_0, \dots, Z_{n+m+1}) = (X_0, \dots, X_{i_1}, Y_0, \dots, Y_{j_1}, X_{i_1+1}, \dots).$$

Then a required t-filtration

$$\begin{tikzcd}[column sep=small]
& Z_0 \arrow[dr, dashed] & & Z_1 \arrow[dr, dashed] & & Z_2 \arrow[dr, dashed] & & \\
X \oplus Y \arrow[ur, "h_0"] & & F^1(X \oplus Y) \arrow[ll] \arrow[ur, "h_1"] & & F^2(X \oplus Y) \arrow[ll] \arrow[ur, "h_2"] & & F^3(X \oplus Y) \arrow[ll] & \cdots \arrow[l]
\end{tikzcd}$$

is constructed as follows:

$$\begin{aligned}
h_0 &= q_0 \oplus 0, & &\dots, & h_{i_1} &= q_{i_1} \oplus 0, \\
F^1(X \oplus Y) &= F^1X \oplus Y, & &\dots, & F^{i_1}(X \oplus Y) &= F^{i_1}X \oplus Y, \\
h_{i_1+1} &= 0 \oplus p_0, & &\dots, & h_{i_1+j_1} &= 0 \oplus p_{j_1}, \\
F^{i_1+1}(X \oplus Y) &= F^{i_1}X \oplus F^1Y, & &\dots, & F^{i_1+j_1}(X \oplus Y) &= F^{i_1}X \oplus F^{j_1}Y
\end{aligned}$$

e. t. c.

Proposition 4.3 is completely proven.

Corollary 4.5. *Let $X \in \mathcal{T}$ be a semistable object of slope φ. If X admits a direct sum decomposition $X = X^1 \oplus X^2 \oplus \cdots \oplus X^k$, then each direct summand X^i is a semistable object of slope φ as well.*

Proof. It is enough to take $k = 2$, $X = X^1 \oplus X^2$. Let

$$X^1 \rightsquigarrow (X^1_{\varphi_0}, X^1_{\varphi_1}, \dots, X^1_{\varphi_n}), \qquad X^2 \rightsquigarrow (X^2_{\psi_0}, X^2_{\psi_1}, \dots, X^2_{\psi_m}).$$

be the Harder–Narasimhan filtrations of the direct summands. By Proposition 4.3, X admits t-filtration whose factors form the increasing shuffle permutation of these semistable factors. Such a filtration is forced to be the Harder–Narasimhan filtration for X. Since X is semistable, its Harder–Narasimhan filtration should be trivial by Theorem 4.1. So, $n = m = 0$ and $\varphi_0 = \psi_0$.

§ 5. Relations between t-stabilities and t-structures

We have seen in Lemma 3.2 that each bounded t-structure on a triangulated category induces some t-stability. On the other hand, we shall show that any t-stability leads to a collection of associated t-structures. To formulate this precisely, we need the following useful notations.

Given a set S of objects of a triangulated category $\mathcal{T}$, write $\langle S \rangle$ for the smallest strictly full extension closed subcategory of $\mathcal{T}$ that contains all objects of S. Note that for any t-stability $(\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})$ on $\mathcal{T}$ and any subset $\Psi \subset \Phi$, the category $\langle \Pi_\varphi \mid \varphi \in \Psi \rangle$ consists precisely of all objects X whose Harder–Narasimhan filtrations $X \rightsquigarrow (X_{\psi_0}, \dots, X_{\psi_k})$ satisfy $\psi_i \in \Psi$ for all i.

Lemma 5.1. *Let $(\Phi, \{\Pi_\varphi\}_{\varphi\in\Phi})$ be a* t*-stability on a triangulated category $\mathcal{T}$. Given an arbitrary decomposition $\Phi = \Phi_- \sqcup \Phi_+$ such that the both $\Phi_\pm$ are non empty and the inequality $\varphi_- < \varphi_+$ holds for each pair $\varphi_- \in \Phi_-$, $\varphi_+ \in \Phi_+$, then a* t*-structure on $\mathcal{T}$ is defined where*

$$\mathcal{T}^{\geqslant 0} = \langle \Pi_\varphi \mid \varphi \in \tau(\Phi_-)\rangle, \qquad \mathcal{T}^{\leqslant 0} = \langle \Pi_\varphi \mid \varphi \in \Phi_+\rangle$$

(and τ above is the non decreasing automorphism of Φ induced by the grading shift functor on $\mathcal{T}$).

Corollary 5.2. *There are two* t*-structures $\mathcal{A}$, $\mathcal{B}$ associated with an arbitrary element $\varphi \in \Phi$ such that*

$$\mathcal{A}_\varphi^{\geqslant 0} = \langle \Pi_\psi \mid \psi \leqslant \tau(\varphi)\rangle, \qquad \mathcal{A}_\varphi^{\leqslant 0} = \langle \Pi_\psi \mid \psi > \varphi\rangle,$$
$$\mathcal{B}_\varphi^{\geqslant 0} = \langle \Pi_\psi \mid \psi < \tau(\varphi)\rangle, \qquad \mathcal{B}_\varphi^{\leqslant 0} = \langle \Pi_\psi \mid \psi \geqslant \varphi\rangle.$$

Proof of Lemma 5.1. By the definition of t-stability, $\Pi_\varphi[-1] = \Pi_{\tau^{-1}(\varphi)}$ and $\tau(\varphi) \geqslant \varphi$. This implies the first axiom of the t-structure:

$$\mathcal{T}^{\leqslant 0} \subset \mathcal{T}^{\leqslant 0}[-1], \qquad \mathcal{T}^{\geqslant 0} \supset \mathcal{T}^{\geqslant 0}[-1].$$

Further, the vanishing condition $\operatorname{Hom}^{\leqslant 0}(\Pi_\varphi, \Pi_\psi) = 0$ for $\varphi > \psi$ implies orthogonality:

$$\operatorname{Hom}^{\leqslant 0}(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0}[-1]) = 0.$$

To verify the third axiom of the t-structure, consider the Harder–Narasimhan filtration of a non zero object $X \in \mathcal{T}$:

$$\begin{tikzcd}
 & X_{\varphi_0} \arrow[dr, dashed] & & X_{\varphi_1} \arrow[dr, dashed] & & & & X_{\varphi_n} \arrow[dr, dashed] & \\
X = F^0X \arrow[ur, "q_0"] & & F^1X \arrow[ll, "p_1"] \arrow[ur, "q_1"] & & F^2X \arrow[ll, "p_2"] & \cdots \arrow[l] & F^nX \arrow[l] \arrow[ur, "q_n"] & & F^{n+1}X = 0. \arrow[ll, "p_{n+1}"]
\end{tikzcd}$$

If $\varphi_n \in \Phi_-$ or $\varphi_0 \in \Phi_+$, then the required triangle is trivial. Otherwise find a number k such that $\varphi_k \in \Phi_-$ but $\varphi_{k+1} \in \Phi_+$ and write $X_{\geqslant 1}$ for a cone of the morphism $X_{\leqslant 0} = F^{k+1}X \xrightarrow{p_{k+1}p_k\cdots p_1} X$. We get the required exact triangle

$$\begin{tikzcd}
 & X_{\geqslant 1} \arrow[dr, dashed] & \\
X \arrow[ur] & & X_{\leqslant 0}, \arrow[ll]
\end{tikzcd}$$

where $X_{\geqslant 1} \in \mathcal{T}^{\geqslant 1}$ and $X_{\leqslant 0} \in \mathcal{T}^{\leqslant 0}$, from Proposition 4.3.

Let us remark that the just constructed t-structure is bounded if and only if

$$\Phi = \Big(\bigcup_{n\in\mathbb{Z}_{\geqslant 0}} \tau^{-n}(\Phi_+)\Big) \quad \text{and} \quad \Phi = \Big(\bigcup_{n\in\mathbb{Z}_{\geqslant 0}} \tau^{n}(\Phi_-)\Big).$$

Let us also stress that different t-stabilities may induce the same t-structure.

Definition 5.3. Let $(\Phi, \{\Pi_\varphi\}_{\varphi\in\Phi})$, $(\Psi, \{P_\psi\}_{\psi\in\Psi})$ be two t-stabilities on a triangulated category $\mathcal{T}$ and let the grading shift functor act on Φ, Ψ by automorphisms τ_Φ, τ_Ψ. We say that t-stability Φ is *finer* than Ψ (or Ψ is *coarser* than Φ) and write $\Phi \preceq \Psi$, if there exists a surjective map $r\colon \Phi \to \Psi$ such that:

1) $r\tau_\Phi = \tau_\Psi r$;
2) $\varphi' > \varphi'' \Leftrightarrow r(\varphi') \geqslant r(\varphi'')$;
3) $\forall\ \psi \in \Psi, \quad P_\psi = \langle \Pi_\varphi \mid \varphi \in r^{-1}(\psi)\rangle$.

In other words, a coarser t-stability is obtained from a finer one by a fusion of some blocks of consequent semistable subcategories. The relation "finer–coarser" defines a partial ordering on the set of all t-stabilities on a given triangulated category. The minimal elements w. r. t. this partial ordering will be called *the finest* t-stabilities. Certainly, the finest t-stabilities accumulate the most complete information about the t-structures.

Proposition 5.4. *Let a triangulated category $\mathcal{T}$ satisfy the following finiteness condition: any* t*-stability Ψ is coarser then some finest* t*-stability $\Phi \preceq \Psi$ (which may depend on Ψ). Then for any bounded* t*-structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ there exists a* t*-stability $\Phi = (\Phi, \{\Pi_\varphi\}_{\varphi\in\Phi})$ and a decomposition $\Phi = \Phi_- \sqcup \Phi_+$ such that*

$$\mathcal{T}^{\leqslant 0} = \langle \Pi_\varphi \mid \varphi \in \Phi_+\rangle, \qquad \mathcal{T}^{\geqslant 0} = \langle \Pi_\varphi \mid \varphi \in \tau(\Phi_-)\rangle$$

(moreover, Φ can be chosen from the set of finest t*-stabilities).*

Proof. This follows at once from Lemmas 3.2, 5.1.

The next proposition gives an easy criterium for compatibility of two t-stabilities as well as a simple sufficient condition for the minimality.

Proposition 5.5. (1) *In the notations of Definition* 5.3 $\Phi \preceq \Psi$ *iff the following three conditions are satisfied:*

(i) *each Φ-semistable object is Ψ-semistable as well;*

(ii) *for any pair of inclusions $\Pi_{\varphi_i} \subset P_{\psi_i}$ (where $i = 1, 2$) the inequality $\varphi_1 < \varphi_2$ implies the inequality $\psi_1 \leqslant \psi_2$;*

(iii) *if $\Pi_\varphi \subset P_\psi$, then $\Pi_{\tau_\Phi(\varphi)} \subset P_{\tau_\Psi(\psi)}$.*

(2) *If the following condition holds for each $\varphi \in \Phi$:*

$$\forall X, Y \in \Pi_\varphi \quad \mathrm{Hom}_{\mathcal{T}}(X, Y) \neq 0 \quad \text{and} \quad \mathrm{Hom}_{\mathcal{T}}(Y, X) \neq 0,$$

then the t*-stability $(\Phi, \{\Pi_\varphi\}_{\varphi\in\Phi}, \tau_\Phi)$ is the finest.*

Proof. If $\Phi \preceq \Psi$, then the conditions (i)–(iii) follow immediately from the definition. To prove the opposite implication, we only need to check that each semistable category P_ψ is spanned by some collection of semistable subcategories Π_φ. Let $X_\psi \rightsquigarrow (X_{\varphi_0}, \dots, X_{\varphi_n})$, where $X_{\varphi_i} \in \Pi_{\varphi_i}$, be the Harder–Narasimhan filtration of an arbitrary $X_\psi \in P_\psi$ w. r. t. Φ. It is enough to show that all $X_{\varphi_i} \in P_\psi$.

By (i), each X_{φ_i} is Ψ-semistable, i. e. $X_{\varphi_i} \in P_{\psi_i}$ for some $\psi_i \in \Psi$. Then $\psi_0 \geqslant \psi$ by Proposition 4.2, because of $\mathrm{Hom}^0(X_\psi, X_{\varphi_0}) \neq 0$ and $X_{\varphi_0} \in P_{\psi_0}$. Similarly, $\psi_n \leqslant \psi$, because of $\mathrm{Hom}^0(X_{\varphi_n}, X_\psi) \neq 0$. But by (ii) $\varphi_0 < \varphi_1 < \dots < \varphi_n$ implies $\psi_0 \leqslant \psi_1 \leqslant \dots \leqslant \psi_n$, which is possible only if $\psi_0 = \psi_n = \psi_i = \psi$.

To prove the assertion (2), just note that if a t-stability $(\Phi, \{\Pi_\varphi\}_{\varphi\in\Phi})$ is not the finest, then there are an element $\varphi \in \Phi$ and some non empty subcategories Π_-,

Π_+ such that $\Pi_\varphi = \langle \Pi_-, \Pi_+ \rangle$ and $\mathrm{Hom}^{\leqslant 0}(\Pi_+, \Pi_-) = 0$ in contradiction to the condition.

Remark 5.6. We will see later that there exist non compatible t-stabilities satisfying the condition (i) from the proposition. So, the other conditions are also essential.

§ 6. t-STABILITIES ON $D^b(\mathcal{C}oh\,\mathbb{P}_1)$

In this paragraph we describe all t-stabilities and bounded t-structures on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ — the bounded derived category of coherent sheaves on projective line. We will identify the coherent sheaves with 0-complexes from $D^b(\mathcal{C}oh\,\mathbb{P}_1)$.

As the first step, we introduce two series of the finest t-stabilities on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$: the standard ones and the exceptional. Then we show that any t-stability on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ is coarser than one of these finest t-stabilities. This result immediately leads to complete list of the bounded t-structures on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$.

6.1. Standard finest t-stabilities. Let $\big(\mathbb{Z}, \{(\mathcal{C}oh\,\mathbb{P}_1)[i]\}_{i\in\mathbb{Z}}\big)$ be the t-stability on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ induced by the tautological t-structure on the derived category as it was explained in Lemma 3.4. The standard finest t-stabilities $(\mathcal{M}, \{\Pi_\mu\}_{\mu\in\mathcal{M}})$ will be obtained from $\big(\mathbb{Z}, \{(\mathcal{C}oh\,\mathbb{P}_1)[i]\}_{i\in\mathbb{Z}}\big)$ by the all possible refinements.

To describe such a refinement precisely, recall that any coherent sheaf F on $\mathbb{P}_1$ is a direct sum of the form

$$F = \Big(\bigoplus_j \Xi_{x_j}\Big) \oplus \Big(\bigoplus_i \mathcal{O}(n_i)\Big),$$

where Ξ_{x_j} is a torsion sheaf with a support at a point x_j and $\mathcal{O}(n_i)$ is a line bundle of degree n_i.

Let $\mathcal{M} = \mathbb{Z} \times (\mathbb{Z} \sqcup \mathbb{P}_1)$. For any $\mu \in \mathcal{M}$ (i. e. for $\mu = (i,x)$ or $\mu = (i,n)$, where $i,n \in \mathbb{Z}$, $x \in \mathbb{P}_1$) define the semistable subcategory Π_μ by prescribing:

$$\Pi_{(i,x)} = \langle \mathcal{O}_x[i] \rangle \qquad \text{or} \qquad \Pi_{(j,n)} = \langle \mathcal{O}(n)[j] \rangle. \tag{6.1}$$

Because of the orthogonality conditions $\mathrm{Hom}^{\leqslant 0}(\Pi_\mu, \Pi_\nu) = 0$ for $\mu > \nu$ any ordering on $\mathcal{M}$ has to contain the following inequalities

$$\begin{array}{lll} (i,\alpha) > (j,\beta) & \text{for} & i > j, \quad \alpha,\beta \in \mathbb{Z}\sqcup\mathbb{P}_1, \\ (i,n) > (i,m) & \text{for} & n > m \in \mathbb{Z}, \quad i \in \mathbb{Z}, \\ (i,x) > (i,n) & \text{for} & x \in \mathbb{P}_1, \quad i,n \in \mathbb{Z}, \end{array}$$

which order all the elements of $\mathcal{M}$ except for the pairs $\big((i,x),(i,y)\big)$. Since $\mathrm{Hom}^q(\mathcal{O}_x[i], \mathcal{O}_y[j]) = 0$ for $x \neq y$ and all $i,j,q \in \mathbb{Z}$, there are no restrictions on the choice of ordering on the set of torsion sheaves.

Fixing any ordering of torsion sheaves, we get the finest t-stability. Indeed, the subcategories Π_μ satisfy the sufficient condition from Proposition 5.5, (2): there is a non zero morphism between any two objects of same Π_μ. The existence of a finite Harder–Narasimhan filtration for any non zero object follows from Proposition 4.3, because of $\mathcal{C}oh\,\mathbb{P}_1[i] = \langle \Pi_{(i,\alpha)} \mid \alpha \in \mathbb{Z}\sqcup\mathbb{P}_1 \rangle$.

Let us stress that t-stabilities $(\mathcal{M}, \{\Pi_\mu\}_{\mu\in\mathcal{M}})$ just constructed depend on an arbitrary choice of the ordering of the $\mathbb{P}_1$-component of the slope. So, they form an infinite family of t-stabilities. In what follows, the name *standard finest* t*-stability* will apply to an arbitrary element of this infinite family.

6.2. Exceptional finest t-stabilities. These t-stabilities are obtained by all possible refinements of the exceptional t-stabilities $\left(\{0,1\},\{\Pi_i^k\}_{i\in\{0,1\}}\right)$ described in Proposition 3.6.

Recall that the subcategories

$$\Pi_0^k = \langle \mathcal{O}(k)[j] \mid j \in \mathbb{Z}\rangle,$$
$$\Pi_1^k = \langle \mathcal{O}(k+1)[i] \mid i \in \mathbb{Z}\rangle$$

are preserved by the grading shift functor and come from the orthogonal decomposition of $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ w. r. t. exceptional base $(\mathcal{O}(k),\mathcal{O}(k+1))$, where k is an arbitrary but fixed integer (so, there is a countable set of the exceptional stabilities parameterized by k).

To describe all the finest refinements of $\left(\{0,1\},\{\Pi_i^k\}_{i\in\{0,1\}}\right)$, let us take

$$\mathcal{E} = \mathbb{Z}\times\{0,1\}$$

as the set of slopes and consider the following semistable subcategories:

$$\Pi^k_{(i,0)} = \langle \mathcal{O}(k)[i]\rangle, \qquad \Pi^k_{(i,1)} = \langle \mathcal{O}(k+1)[i]\rangle. \tag{6.2}$$

The grading shift functor acts on them by an automorphism $\tau\in\operatorname{Aut}\mathcal{E}$ that takes $\tau(i,0)=(i+1,0)$, $\tau(i,1)=(i+1,1)$. These categories, clearly, satisfy the sufficient condition from Proposition 5.5, 2) (there is some non zero morphism between any two objects of the same category). So, to convert them into the finest t-stability, it remains to equip them with an appropriate ordering and verify the existence of finite Harder–Narasimhan filtration with the factors from Π_ε for any object $X\in D^b(\mathcal{C}oh\,\mathbb{P}_1)$.

To this aim, we apply the following well known result to $D^b(\mathcal{C}oh\,\mathbb{P}_1)$.

Proposition 6.1. *Let $\mathcal{A}$ be an abelian category of homological dimension 1. Then each object $X\in D^b(\mathcal{A})$ splits into finite direct sum of i-complexes:*

$$X = \bigoplus_i A_i[-i],$$

where $A_i\in\mathcal{A}$ and $A_i[-i]$ is a complex with only one non zero term $A_i\in\mathcal{A}$, which is situated at i-th place.

It implies that any non zero object of $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ is a direct sum of invertible and torsion sheaves possibly shifted through the derived category grading. Thus, it is enough to construct the Harder–Narasimhan filtrations only for invertible sheaves $\mathcal{O}(n)$ and torsion sheaves Ξ_x with one point support x. These filtrations are

$$\begin{tikzcd} & \mathcal{O}(k)^{\oplus(n-k-1)}[1] \arrow[dr, dashed] & \\ \mathcal{O}(n) \arrow[ur] & & \mathcal{O}(k+1)^{\oplus(n-k)}, \arrow[ll] \end{tikzcd} \tag{6.3}$$

if $n>k+1$, and

$$\begin{tikzcd} & \mathcal{O}(k)^{\oplus(k-n+1)} \arrow[dr, dashed] & \\ \mathcal{O}(n) \arrow[ur] & & \mathcal{O}(k+1)^{\oplus(k-n)}[-1], \arrow[ll] \end{tikzcd} \tag{6.4}$$

if $n<k$, and

$$\begin{tikzcd} & \mathcal{O}(k)^{\oplus d}[1] \arrow[dr, dashed] & \\ \Xi_x \arrow[ur] & & \mathcal{O}(k+1)^{\oplus d}, \arrow[ll] \end{tikzcd} \tag{6.5}$$

where $d = \deg \Xi_x$. They are obtained from the following canonical exact triples in $\mathcal{C}oh\, \mathbb{P}_1$:

$$\begin{aligned} &0 \to \mathcal{O}(k)^{\oplus(n-k-1)} \to \mathcal{O}(k+1)^{\oplus(n-k)} \to \mathcal{O}(n) \to 0, && \text{if} \quad n > k+1, \\ &0 \to \mathcal{O}(n) \to \mathcal{O}(k)^{\oplus(k-n+1)} \to \mathcal{O}(k+1)^{\oplus(k-n)} \to 0, && \text{if} \quad n < k, \\ &0 \to \mathcal{O}(k)^{\oplus d} \to \mathcal{O}(k+1)^{\oplus d} \to \Xi_x \to 0, && \text{where} \quad d = \deg \Xi_x. \end{aligned}$$

Any ordering on $\mathcal{E}$ comparable with the orthogonality conditions $\operatorname{Hom}^{\leqslant 0}\left(\Pi^k_{\varepsilon'}, \Pi^k_{\varepsilon''}\right) = 0$ for $\varepsilon'' < \varepsilon'$ should contain inequalities $(i,0) < (j,0)$, $(i,1) < (j,1)$ for all $i < j$. Further, the tower (6.4) is a Harder–Narasimhan filtration only if $\mathcal{O}(k) < \mathcal{O}(k+1)[-1]$. This forces inequalities $(i,0) < (i-1,1)$ for all $i \in \mathbb{Z}$. It is easy to see that a linear ordering on $\mathcal{E}$ extending these three sets of inequalities is described by a number $p \in \mathbb{N} \cup \{0, +\infty\}$ and looks like

$$\begin{aligned} &\cdots < (i+p,0) < (i-1,1) < (i+p+1,0) < (i+p,1) < \cdots, && p \neq \infty, \\ &(i-1,0) < (i,0) < (j,1) < (j+1,1) \quad \forall i,j, && p = \infty. \end{aligned} \tag{6.6}$$

We write $\mathcal{E}_p$ for the set $\mathcal{E}$ linearly ordered according to (6.6).

Thus, for any $k \in \mathbb{Z}$, $p \in \mathbb{N} \cup \{0, +\infty\}$ we have a finest t-stability $\left(\mathcal{E}_p, \{\Pi^k_\varepsilon\}_{\varepsilon \in \mathcal{E}_p}\right)$. We call them *exceptional finest* t*-stabilities*. All exceptional finest t-stabilities with the same k have the same collection of semistable subcategories and differ from each other only by the ordering of their slopes.

6.3. Description of all t-stabilities on $D^b(\mathcal{C}oh\, \mathbb{P}_1)$. In this section we establish the following classification:

Theorem 6.2. *All finest* t*-stabiliton $D^b(\mathcal{C}oh\, \mathbb{P}_1)$ are exhausted by the standard and exceptional finest* t*-stabilities described in sections* 6.1, 6.2*. All other* t*-stabilities are obtained by coarsening the finest ones.*

Proof. Let t-stability $(\Phi, \{\Theta_\varphi\}_{\varphi \in \Phi})$ be non compatible with any standard finest t-stability on $D^b(\mathcal{C}oh\, \mathbb{P}_1)$. We are going to show that it should be coarser or equal to some exceptional finest t-stability.

Since Φ can not be refined into a standard t-stability $\mathcal{M}$, at least one of the standard semistable subcategories Π_μ is generated by an object X that is not $\mathcal{M}$-semistable, otherwise all Π_μ would be $\mathcal{M}$-semistable and Φ would be coarser than $\mathcal{M}$. Moreover, since the grading shift takes semistable objects to semistable ones, we can suppose without loss of generality that X is either $\mathcal{O}_x$ or $\mathcal{O}(n)$.

Let

$$\begin{tikzcd} & X_{\varphi_0} \arrow[dr, dashed] & \\ X \arrow[ur, "\wr"] & & F^1X. \arrow[ll] \end{tikzcd} \tag{6.7}$$

be the first exact triangle of the Harder–Narasimhan filtration of X w. r. t. Φ. The next lemma plays the central role in our proof.

Lemma 6.3. *Let $X \in D^b(\mathcal{C}oh\,\mathbb{P}_1)$ coincide with either O_x or $\mathcal{O}(n)$. If*

$$\mathrm{Hom}^{\leqslant 0}(F^1X, X_{\varphi_0}) = 0$$

in the exact triangle (6.7)*, then this triangle is isomorphic to one of the triangles* (6.3)–(6.5) *for some $k \in \mathbb{Z}$.*

We will prove this lemma later, and now let us continue with the proof of the theorem. By the lemma, the first Harder–Narasimhan factor of X w. r. t. Φ is either $\mathcal{O}(k)^{\oplus n_0}$ or $\mathcal{O}(k)^{\oplus n_0}[1]$ for some $n_0 \in \mathbb{N}$ and $k \in \mathbb{Z}$. Since it is Φ-semistable, all the objects $\mathcal{O}(k)[i]$ are Φ-semistable for all $i \in \mathbb{Z}$. Hence, the exact triangle (6.7) has the form

$$\begin{tikzcd} & \mathcal{O}(k)^{\oplus n_0}[j] \arrow[dr, dashed] & \\ X \arrow[ur] & & \mathcal{O}(k+1)^{\oplus m_1}[j-1], \arrow[ll] \end{tikzcd} \tag{6.8}$$

where $n_0, m_1 \in \mathbb{N}$ and $j = 0, 1$.

Let us show that the object $\mathcal{O}(k+1)^{\oplus m_1}[j-1]$ is Φ-semistable as well. Indeed, in the contrary case, the Harder–Narasimhan filtration for X w. r. t. Φ can be continued as

$$\begin{tikzcd} & \mathcal{O}(k)^{\oplus n_0}[j] \arrow[dr, dashed] & & X_{\varphi_1} \arrow[dr, dashed] & \\ X \arrow[ur] & & \mathcal{O}(k+1)^{\oplus m_1}[j-1] \arrow[ll] \arrow[ur] & & F^2X. \arrow[ll] \end{tikzcd}$$

It follows from Proposition 4.2 and the definition of the Harder–Narasimhan filtration that

$$\mathrm{Hom}^{\leqslant 0}(F^2X,\, X_{\varphi_1}) = 0, \tag{6.9}$$

$$\mathrm{Hom}^{\leqslant 0}(X_{\varphi_1},\, \mathcal{O}(k)^{\oplus n_0}[j]) = 0. \tag{6.10}$$

Applying Lemma 6.3 once more, we see that for some $m \in \mathbb{Z}$ we have either $X_{\varphi_1} = \mathcal{O}(m)^{\oplus n_1}[j-1]$, $F^2X = \mathcal{O}(m+1)^{\oplus m_2}[j-2]$ (here $k < m-1$) or $X_{\varphi_1} = \mathcal{O}(m)^{\oplus n_1}[j]$, $F^2X = \mathcal{O}(m+1)^{\oplus m_2}[j-1]$ for some $n_1, m_2 \in \mathbb{N}$ (here $k > m$). In the first case

$$\mathrm{Hom}^0\big(X_{\varphi_1},\, \mathcal{O}(k)^{\oplus n_0}[j]\big) = \mathrm{Ext}^1_{\mathcal{C}oh\,\mathbb{P}_1}\big(\mathcal{O}(m)^{\oplus n_1},\, \mathcal{O}(k)^{\oplus n_0}\big) \neq 0,$$

which contradicts to (6.10). In the second case

$$\mathrm{Hom}^0\big(X_{\varphi_1},\, \mathcal{O}(k)^{\oplus n_0}[j]\big) = \mathrm{Hom}_{\mathcal{C}oh\,\mathbb{P}_1}\big(\mathcal{O}(m)^{\oplus n_1},\, \mathcal{O}(k)^{\oplus n_0}\big) \neq 0,$$

which contradicts to (6.10) as well.

So, the Harder–Narasimhan filtration (6.7) can not be continued, i. e. the object $\mathcal{O}(k+1)^{\oplus m_1}[j-1]$ is Φ-semistable. Hence, all the objects $\mathcal{O}(k+1)[j]$ are Φ-semistable as well for all $j \in \mathbb{Z}$.

Since for our fixed k all $\mathcal{O}(k)[i]$ and $\mathcal{O}(k+1)[i]$ are Φ-semistable, we can assume (maybe after appropriate refinement of Φ) that subcategories $\langle \mathcal{O}(k)[i] \rangle$ and $\langle \mathcal{O}(k+1)[i] \rangle$ are Φ-semistable. Let us show that any Φ-semistable object Y should belong to one of these subcategories. To this aim we represent Y by a direct sum of the shifts of invertible sheaves and one point skyscrapers. The Harder–Narasimhan factors of such a summand (w. r. t. Φ) are exhausted by the objects $\mathcal{O}(k)[i]$ and $\mathcal{O}(k+1)[i]$ (see (6.3), (6.4)). It follows from Proposition 4.3 and the uniqueness of a Harder–Narasimhan filtration that all Harder–Narasimhan factors of Y are exhausted by $\mathcal{O}(k)^{\oplus a}[i]$ and $\mathcal{O}(k+1)^{\oplus b}[i]$ as well.

Thus, the collection of Φ-semistable subcategories coincides with the collection of semistable subcategories for one of the finest exceptional t-stabilities. This determines unequally the action of the grading shift functor on Φ. The theorem is completely proven.

Proof of Lemma 6.3. Let us start with the following evident remark:

Remark 6.4. Let $X_{\varphi_0} = X_1 \oplus X_2$ and $f = f_1 \oplus f_2$ (where $f_i \in \mathrm{Hom}(X, X_i)$) in an exact triangle (6.7). If $f_2 = 0$, then $F^1X = Y \oplus X_2[-1]$ and $\mathrm{Hom}^{\leqslant 0}(F^1X, X_{\varphi_0}) \neq 0$.

Now, let in the triangle (6.7) X be one of the two 0-complexes: $\mathcal{O}_x[0]$ or $\mathcal{O}(n)[0]$. By Proposition 6.1 and the previous remark, the vanishing $\mathrm{Hom}^{\leqslant 0}(F^1X, X_{\varphi_0}) = 0$ implies $X_{\varphi_0} = X_0[0] \oplus X_1[1]$ and $f = f_0 \oplus f_1$, where

$$X_i \in \mathcal{C}oh\,\mathbb{P}_1\,, \quad f_0 \in \mathrm{Hom}_{\mathcal{C}oh\,\mathbb{P}_1}(X, X_0)\,, \quad f_1 \in \mathrm{Ext}^1_{\mathcal{C}oh\,\mathbb{P}_1}(X, X_1).$$

Moreover, $f_i = 0 \Leftrightarrow X_i = 0$.

Let us write C for the cone of the morphism f. Then $F^1X = C[-1]$ and the element $f_1 \in \mathrm{Ext}^1_{\mathcal{C}oh\,\mathbb{P}_1}(X, X_1)$ gives an extension

$$0 \to X_1 \to E \xrightarrow{d_{-1}} X \to 0. \tag{6.11}$$

Thus $X_1[1] \in D^b(\mathcal{C}oh\,\mathbb{P}_1)$ can be represented by a complex

$$0 \to E \xrightarrow{d_{-1}} X \to 0,$$

and f_1 by a morphism of complexes

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\mathrm{Id}_X"] & 0 \arrow[d] \\
0 \arrow[r] & E \arrow[r, "d_{-1}"] & X \arrow[r] & 0\,.
\end{tikzcd}$$

We can represent the morphism $f\colon X \to X_0[0] \oplus X_1[1]$ by the following morphism of complexes:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\mathrm{Id}_X \oplus f_0"] & 0 \arrow[d] \\
0 \arrow[r] & E \arrow[r, "d_{-1}\oplus 0"] & X \oplus X_0 \arrow[r] & 0\,.
\end{tikzcd}$$

So, the cone C is represented by a complex

$$0 \to X \oplus E \xrightarrow{\delta_{-1}} X \oplus X_0 \to 0,$$

where δ_{-1} has a matrix

$$\delta_{-1} = \begin{pmatrix} \mathrm{Id}_X & d_{-1} \\ f_0 & 0 \end{pmatrix}.$$

In the derived category $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ the cone C splits as $(\operatorname{Ker}\delta_{-1})[1]\oplus(\operatorname{Coker}\delta_{-1})[0]$. In the abelian category $\mathcal{C}oh\,\mathbb{P}_1$ we have the following commutative diagram $\mathcal{C}oh\,\mathbb{P}_1$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & E \arrow[r, "d_{-1}"] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker}\delta_{-1} \arrow[r] \arrow[d] & X\oplus E \arrow[r, "\delta_{-1}"] \arrow[d] & X\oplus X_0 \arrow[r] \arrow[d] & \operatorname{Coker}\delta_{-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker} f_0 \arrow[r] \arrow[d] & X \arrow[r, "f_0"] \arrow[d] & X_0 \arrow[r] \arrow[d] & \operatorname{Coker} f_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

which shows that $\operatorname{Coker}\delta_{-1} = \operatorname{Coker} f_0$, and we get

$$\operatorname{Hom}^{\leqslant 0}\big((\operatorname{Ker}\delta_{-1})[0]\oplus(\operatorname{Coker} f_0)[-1],\ X_0[0]\oplus X_1[1]\big) = 0,$$

because of $\operatorname{Hom}^{\leqslant 0}(F^1X,\, X_{\varphi_0}) = 0$. This implies, in particular, that

$$\operatorname{Hom}^{\leqslant 0}\big((\operatorname{Ker}\delta_{-1})[0],\ X_0[0]\big) = 0, \tag{6.12}$$

$$\operatorname{Hom}^{\leqslant 0}\big((\operatorname{Ker}\delta_{-1})[0],\ X_1[1]\big) = 0. \tag{6.13}$$

Firstly, let us assume that $X = \mathcal{O}_x$ and $X_0 \neq 0$. Then $\operatorname{Ker} f_0 = 0$, that is $\operatorname{Ker}\delta_{-1} = X_1$, which contradicts to (6.13). Hence, $X_0 = 0$ the triangle (6.7) comes from the exact sequence (6.11). In particular, $C[-1] = E$, the vanishing $\operatorname{Hom}^{\leqslant 0}(F^1X, X_{\varphi_0}) = 0$ means that $\operatorname{Hom}^{\leqslant 0}(E, X_1[1]) = 0$, i. e. $\operatorname{Hom}_{\mathcal{C}oh\,\mathbb{P}_1}(E, X_1) = 0$ and $\operatorname{Ext}^1_{\mathcal{C}oh\,\mathbb{P}_1}(E, X_1) = 0$. Since $\operatorname{Hom}_{\mathcal{C}oh\,\mathbb{P}_1}(X_1, E) \neq 0$ by the construction, the assertion of Lemma 6.3 follows in the case $X = \mathcal{O}_x$ from the following obvious fact:

Claim 6.5. *Let sheaves E, X_1 on $\mathbb{P}_1$ satisfy the conditions*

$$\operatorname{Hom}_{\mathcal{C}oh\,\mathbb{P}_1}(E, X_1) = 0, \qquad \operatorname{Ext}^1_{\mathcal{C}oh\,\mathbb{P}_1}(E, X_1) = 0, \qquad \operatorname{Hom}_{\mathcal{C}oh\,\mathbb{P}_1}(X_1, E) \neq 0.$$

Then $E = \mathcal{O}(k+1)^{\oplus e}$ and $X_1 = \mathcal{O}(k)^{\oplus x_1}$ for some $k \in \mathbb{Z}$, $e, x_1 \in \mathbb{N}$.

The proof is evident.

Now, let $X = \mathcal{O}(n)$. If the both sheaves X_0, X_1 are non zero, then X_1 contains some invertible sheaf L as a direct summand (otherwise $f_1 = 0$ and $X_1 = 0$, see Remark 6.4). Hence, $\operatorname{Ker}\delta_{-1}$ also contains some invertible sheaf L' as a direct summand (look at the left column of the above diagram). If $\operatorname{Ker} f_0 = 0$, then $\operatorname{Ker}\delta_{-1} = X_1$, which contradicts to (6.13). Thus, $\operatorname{Ker} f_0 \neq 0$ and $\operatorname{Im} f_0$ is a torsion sheaf, i. e. X_0 also should have a non zero torsion $T \subset X_0$. Since $\operatorname{Hom}_{\mathcal{C}oh\,\mathbb{P}_1}(L', T) = 0$, we come to a contradiction with (6.12).

So, for $X = \mathcal{O}(n)$ we have either $X_1 = 0$ or $X_0 = 0$.

If $X_1 = 0$, then the triangle (6.7) comes from the exact sequence

$$0 \to \operatorname{Ker} f_0 \to \mathcal{O}(n) \xrightarrow{f_0} X_0 \to \operatorname{Coker} f_0 \to 0$$

and looks like

(6.14)
$$\begin{tikzcd} & X_0[0] \arrow[dr, dashed] & \\ \mathcal{O}(n) \arrow[ur, "f_0"] & & (\operatorname{Ker} f_0)[0] \oplus (\operatorname{Coker} f_0)[-1]. \arrow[ll] \end{tikzcd}$$

The vanishing condition $\operatorname{Hom}^{\leqslant 0}(F^1X, X_{\varphi_0}) = 0$ can be rewritten as:

$$\operatorname{Hom}^{\leqslant 0}((\operatorname{Ker} f_0)[0],\ X_0[0]) = 0, \tag{6.15}$$

$$\operatorname{Hom}^{\leqslant 0}((\operatorname{Coker} f_0)[-1],\ X_0[0]) = 0. \tag{6.16}$$

Since $\operatorname{Ker} f_0 \neq 0$ contradicts with (6.15) as before, we have $\operatorname{Ker} f_0 = 0$ and it follows from (6.16) and Remark 6.4 that the triangle (6.14) is isomorphic to (6.4) for some integer $k > n$.

Finally, if $X_0 = 0$, then the triangle (6.7) comes from the exact triple

$$0 \to X_1 \to E \xrightarrow{d_{-1}} \mathcal{O}(n) \to 0$$

and looks like

(6.17)
$$\begin{tikzcd} & X_1[1] \arrow[dr, dashed] & \\ \mathcal{O}(n) \arrow[ur, "f_1"] & & E, \arrow[ll] \end{tikzcd}$$

where $\operatorname{Hom}^{\leqslant 0}(E, X_1[1]) = 0$. Using Remark 6.4 once again, we see that the triangle (6.17) is isomorphic to the triangle (6.3) for some integer $k < n - 1$. Lemma 6.3 is completely proven.

6.4. Classification of t-structures on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$. In this section we will list all representatives of bounded t-structures on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ modulo the natural action of the autoequivalence group $\operatorname{Aut}(D^b(\mathcal{C}oh\,\mathbb{P}_1))$ (by [4], this group is generated by the grading shift functor, tensoring by invertible sheaves $\mathcal{O}(n) \in \operatorname{Pic}\mathbb{P}_1$, and linear fractional automorphisms of $\mathbb{P}_1$). Moreover, since the both parts of a t-structure $(D^{\geqslant 0}, D^{\leqslant 0})$ are uniquely recovered from any one of them, we will present the both parts only when such recovering would be not completely evident.

The complete list of all t-stabilities on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ obtained above allows us to describe all bounded t-structures on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ via Lemma 5.1: each t-structure is obtained from some t-stability by splitting the set of semistable categories in two ordered parts and fusing subcategories for each part into one subcategory.

The first part of t-structures on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ comes from the standard finest t-stabilities $(\mathcal{M}, \{\Pi_\mu\}_{\mu\in\mathcal{M}})$ described in Section 6.1. Let us start with the tautological t-structure

$$\mathcal{A}^{\leqslant 0} = \langle (\mathcal{C}oh\,\mathbb{P}_1)[j],\ j \geqslant 0 \rangle\,,$$

whose core is[11] $\mathcal{A}^0 = \mathcal{C}oh\,\mathbb{P}_1$. We depict this schematically as follows:

$$\mathcal{A}^{\geqslant 0} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{A}^{\leqslant 0}$$

$$\dots\,\mathcal{O}[-1]\,\dots\underbrace{\dots\,\mathcal{O}(-1)[0]\,\mathcal{O}[0]\,\mathcal{O}(1)[0]\,\dots\,\mathcal{O}_x[0]\,\dots}_{\mathcal{C}oh\,\mathbb{P}_1[0]}\dots\,\mathcal{O}(-1)[1]\,\mathcal{O}[1]\,\dots$$

To describe the next three t-structures, the langauge of torsion pairs is quite convenient. Let us recall briefly some results from the corresponding theory (see [12], [5]). Given an abelian category $\mathcal{A}$, a pair of its full subcategories $(\mathcal{A}_1, \mathcal{A}_0)$ is called a *torsion pair*, if $\operatorname{Hom}(\mathcal{A}_1, \mathcal{A}_0) = 0$ and any object $A \in \mathcal{A}$ is fitted in a unique exact triple

$$0 \to A_1 \to A \to A_0 \to 0,$$

where $A_i \in \mathcal{A}_i$. A torsion pair $(\mathcal{A}_1, \mathcal{A}_0)$ is called *cotilting*, if each object of $\mathcal{A}$ can be represented as a factor object of some object from $\mathcal{A}_0$.

Any torsion pair $(\mathcal{A}_1, \mathcal{A}_0)$ gives a t-structure on $D^b(\mathcal{A})$ by the following prescription:

$$\begin{aligned} {}^pD^b(\mathcal{A})^{\leqslant 0} &= \bigl\{A \in D^b(\mathcal{A})^{\leqslant 0} \mid H^0(A) \in \mathcal{A}_1\bigr\}, \\ {}^pD^b(\mathcal{A})^{\geqslant 0} &= \bigl\{A \in D^b(\mathcal{A})^{\geqslant -1} \mid H^{-1}(A) \in \mathcal{A}_0\bigr\}. \end{aligned}$$

The core of this t-structure is called *the cotilt* of $\mathcal{A}$ w. r. t. $(\mathcal{A}_1, \mathcal{A}_0)$ and is denoted by ${}^p\mathcal{A}$. If a torsion pair $(\mathcal{A}_1, \mathcal{A}_0)$ is cotilting, then there is a canonical equivalence of the derived categories $D({}^p\mathcal{A}) = D(\mathcal{A})$.

There are two cotilting pairs $\mathcal{B}$, $\mathcal{C}$ in $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ depicted (together with the corresponding cotilts) as follows:

$$\begin{aligned} \mathcal{B}_0 &= \langle \mathcal{O}(n),\ n < 0\rangle, \qquad \mathcal{B}_1 = \langle \mathcal{O}(n),\ n \geqslant 0;\ \mathcal{O}_x,\ x \in \mathbb{P}_1\rangle, \\ \mathcal{B}^{\leqslant 0} &= \langle \mathcal{O}(n)[i],\ n \geqslant 0,\ i \geqslant 0;\ \mathcal{O}_x[i],\ x \in \mathbb{P}_1,\ i \geqslant 0;\ \mathcal{O}(n)[j],\ n < 0,\ j \geqslant 1\rangle, \\ \mathcal{B}^0 &= \langle \mathcal{O}(n)[0],\ n \geqslant 0;\ \mathcal{O}_x[0],\ x \in \mathbb{P}_1;\ \mathcal{O}(n)[1],\ n < 0\rangle, \end{aligned}$$

$$\mathcal{B}^{\geqslant 0} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{B}^{\leqslant 0}$$

$$\dots\,\mathcal{O}[-1]\,\dots\underbrace{\dots\,\mathcal{O}(-1)[0]\,\mathcal{O}[0]\,\mathcal{O}(1)[0]\dots\,\mathcal{O}_x[0]\,\dots}_{\mathcal{C}oh\,\mathbb{P}_1[0]}\dots\,\mathcal{O}(-1)[1]\,\mathcal{O}[1]\,\dots$$

$$\begin{aligned} \mathcal{C}_0 &= \langle \mathcal{O}(n),\ n \in \mathbb{Z}\rangle, \qquad \mathcal{C}_1 = \langle \mathcal{O}_x,\ x \in \mathbb{P}_1\rangle, \\ \mathcal{C}^{\leqslant 0} &= \langle \mathcal{O}_x[i],\ x \in \mathbb{P}_1,\ i \geqslant 0;\ \mathcal{O}(n)[j],\ n \in \mathbb{Z},\ j \geqslant 1\rangle, \\ \mathcal{C}^0 &= \langle \mathcal{O}_x[0],\ x \in \mathbb{P}_1;\ \mathcal{O}(n)[1],\ n \in \mathbb{Z}\rangle, \end{aligned}$$

[11] here and below we write D^0 for the core of t-structure $(D^{\geqslant 0}, D^{\leqslant 0})$

$$\mathcal{C}^{\geqslant 0} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{C}^{\leqslant 0}$$

$$\dots \mathcal{O}(-1)[0]\,\mathcal{O}[0]\dots \underbrace{\dots \mathcal{O}_x[0]\dots}\dots \mathcal{O}(-1)[1]\,\mathcal{O}[1]\dots \underbrace{\dots \mathcal{O}_x[1]\dots}\dots$$

Finally, there is a family of cotilting pairs $\mathcal{D}(P)$ on $\mathcal{C}oh\,\mathbb{P}_1$ depending on a choice of an *arbitrary*[12] nonempty subset $P \subset \mathbb{P}_1$:

$$\begin{aligned}
\mathcal{D}(P)_0 &= \langle \mathcal{O}_x,\ x \in P\rangle, \qquad \mathcal{D}(P)_1 = \langle \mathcal{O}_y,\ y \notin P;\ \mathcal{O}(n),\ n \in \mathbb{Z}\rangle,\\
\mathcal{D}(P)^{\leqslant 0} &= \langle \mathcal{O}_x[i],\ x \in P,\ i \geqslant 0;\ X[j],\ X \in \mathcal{C}oh\,\mathbb{P}_1,\ j \geqslant 1\rangle,\\
\mathcal{D}(P)^0 &= \langle \mathcal{O}_x[0],\ x \in P;\ \mathcal{O}_y[1],\ y \notin P;\ \mathcal{O}(n)[1],\ n \in \mathbb{Z}\rangle,
\end{aligned}$$

$$\mathcal{D}(P)^{\geqslant 0} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{D}(P)^{\leqslant 0}$$

$$\dots \underbrace{\dots \mathcal{O}_y[0]\dots}_{y \in \mathbb{P}_1 \setminus P} \underbrace{\dots \mathcal{O}_x[0]\dots}_{x \in P} \dots \mathcal{O}[1]\dots \underbrace{\dots \mathcal{O}_y[1]\dots}_{y \in \mathbb{P}_1 \setminus P} \underbrace{\dots \mathcal{O}_x[1]\dots}_{x \in P} \dots$$

Of course, two such t-structures are congruent $\mathcal{D}(P) \sim \mathcal{D}(P')$ modulo the action of autoequivalences $\operatorname{Aut} D^b(\mathcal{C}oh\,\mathbb{P}_1)$ if and only if $P' = \varphi(P)$ for some transformation $\varphi \in \operatorname{Aut}\mathbb{P}_1$.

The second part of t-structures comes from the exceptional finest t-stabilities $\left(\mathcal{E}_p, \{\Pi^0_\varepsilon\}_{\varepsilon \in \mathcal{E}_p}\right)$ described in Section 6.2 (recall that $p \in \overline{\mathbb{N}}$ prescribes an ordering on the set $\mathcal{E} = \mathbb{Z} \times \{0,1\}$). These t-structures are subdivided into bounded t-structures $\mathcal{E}(p)$ and $\mathcal{F}(p)$ and unbounded t-structures $\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$. They are depicted as follows:

$$\begin{aligned}
\mathcal{E}(p)^{\leqslant 0} &= \langle \mathcal{O}[i],\ i \geqslant p;\ \mathcal{O}(1)[j],\ j \geqslant -2\rangle,\\
\mathcal{E}(p)^0 &= \langle \mathcal{O}(p)[0],\ \mathcal{O}(1)[-2]\rangle,
\end{aligned}$$

where p is non negative integer, and

$$\mathcal{E}(p)^{\leqslant 0}$$

$$\dots \mathcal{O}[p-1]\,\mathcal{O}(1)[-2]\,\mathcal{O}[p]\,\mathcal{O}(1)[-1]\,\mathcal{O}[p+1]\,\mathcal{O}(1)[0]\dots$$

$$\mathcal{E}(p)^{\geqslant 0}$$

Further

$$\begin{aligned}
\mathcal{F}(p)^{\leqslant 0} &= \langle \mathcal{O}[i],\ i \geqslant p;\ \mathcal{O}(1)[j],\ j \geqslant -1\rangle,\\
\mathcal{F}(p)^0 &= \langle \mathcal{O}(p)[0],\ \mathcal{O}(1)[-1]\rangle,
\end{aligned}$$

[12] in the set theoretic sense!

where p is non negative integer, and

$$\mathcal{F}(p)^{\leqslant 0}$$

$$\ldots\ \mathcal{O}[p-1]\ \mathcal{O}(1)[-2]\ \mathcal{O}[p]\ \mathcal{O}(1)[-1]\ \mathcal{O}[p+1]\ \mathcal{O}(1)[0]\ \ldots$$

$$\mathcal{F}(p)^{\geqslant 0}$$

$$\begin{aligned}
\mathcal{G}^{\leqslant 0} &= \langle \mathcal{O}[i],\ i \geqslant 0;\ \mathcal{O}(1)[j],\ j \in \mathbb{Z} \rangle, \\
\mathcal{G}^{\geqslant 0} &= \langle \mathcal{O}[i],\ i \leqslant 0 \rangle, \\
\mathcal{G}^{0} &= \langle \mathcal{O}[0] \rangle,
\end{aligned}$$

$$\mathcal{G}^{\geqslant 0} \qquad\qquad\qquad\qquad \mathcal{G}^{\leqslant 0}$$

$$\underbrace{\ldots\ \mathcal{O}[-1]\ \mathcal{O}[0]\ \mathcal{O}[1]\ \ldots}_{\mathcal{O}[j],\ j\in\mathbb{Z}}\ \underbrace{\ldots\mathcal{O}(1)[-1]\ \mathcal{O}(1)[0]\ \mathcal{O}(1)[1]\ \ldots}_{\mathcal{O}(1)[j],\ j\in\mathbb{Z}}$$

$$\begin{aligned}
\mathcal{H}^{\leqslant 0} &= \langle \mathcal{O}(1)[j],\ j \geqslant 0 \rangle, \\
\mathcal{H}^{\geqslant 0} &= \langle \mathcal{O}[i],\ i \in \mathbb{Z};\ \mathcal{O}(1)[j],\ j \leqslant 0 \rangle, \\
\mathcal{H}^{0} &= \langle \mathcal{O}(1)[0] \rangle,
\end{aligned}$$

$$\mathcal{H}^{\geqslant 0} \qquad\qquad\qquad\qquad \mathcal{H}^{\leqslant 0}$$

$$\underbrace{\ldots\ \mathcal{O}[-1]\ \mathcal{O}[0]\ \mathcal{O}[1]\ \ldots}_{\mathcal{O}[j],\ j\in\mathbb{Z}}\ \underbrace{\ldots\mathcal{O}(1)[-1]\ \mathcal{O}(1)[0]\ \mathcal{O}(1)[1]\ \ldots}_{\mathcal{O}(1)[j],\ j\in\mathbb{Z}}$$

$$\begin{aligned}
\mathcal{I}^{\leqslant 0} &= \langle \mathcal{O}(1)[j],\ j \in \mathbb{Z} \rangle, \\
\mathcal{I}^{\geqslant 0} &= \langle \mathcal{O}[i],\ i \in \mathbb{Z} \rangle, \\
\mathcal{I}^{0} &= \langle 0 \rangle,
\end{aligned}$$

$$\mathcal{I}^{\geqslant 0} \qquad\qquad\qquad\qquad \mathcal{I}^{\leqslant 0}$$

$$\underbrace{\ldots\ \mathcal{O}[-1]\ \mathcal{O}[0]\ \mathcal{O}[1]\ \ldots}_{\mathcal{O}[j],\ j\in\mathbb{Z}}\ \underbrace{\ldots\mathcal{O}(1)[-1]\ \mathcal{O}(1)[0]\ \mathcal{O}(1)[1]\ \ldots}_{\mathcal{O}(1)[j],\ j\in\mathbb{Z}}$$

Remark 6.6. It is easy to see that the cores of the exceptional t-structures can be described as follows:

1) $\mathcal{E}(p)^0$ for $p \geqslant 0$ and $\mathcal{F}(p)^0$ for $p \geqslant 1$ are equivalent to a direct sum of two copies of the category of vector spaces $\mathcal{V}ect \oplus \mathcal{V}ect$;

2) $\mathcal{F}(0)^0$ is equivalent to the representation category of the quiver $\bullet\!\rightrightarrows\!\bullet$;

3) $\mathcal{G}^0 \simeq \mathcal{H}^0 \simeq \mathcal{V}ect$;

4) $\mathcal{I}^0 \simeq 0$.

Since the category $\mathcal{V}ect$ is semisimple, the derived category of the core of an exceptional t-structure coincides with the initial derived category $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ only for the structure $\mathcal{F}(0)$ (comp. with [3]).

Because a bounded t-structure on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ induces (by Lemma 3.2) a t-stability whose finest refinement coincides with either $\mathcal{M}$ or $\mathcal{E}$, any bounded t-structure on $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ can be obtained (via Proposition 5.4) from $\mathcal{M}$ or $\mathcal{E}$ in the way just described above. So, we get the following theorem.

Theorem 6.7. *All bounded* t*-structures on* $D^b(\mathcal{C}oh\,\mathbb{P}_1)$ *are exhausted (modulo the action of* $\operatorname{Aut} D^b(\mathcal{C}oh\,\mathbb{P}_1)$*) by the structures* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}(p)$, $\mathcal{F}(p)$ *described above.*

§ 7. t-STABILITIES ON A CURVE OF POSITIVE GENUS

Let $\mathcal{T}_g = D^b(\mathcal{C}oh\,C_g)$ be the bounded derived category of coherent sheaves on a smooth curve of genus g. In this paragraph we study t-stabilities on $\mathcal{T}_g$ as follows. Firstly, we describe a set of the standard finest t-stabilities, which extend the standard Mumford–Takemoto slope on $\mathcal{C}oh\,C_g$. Then we show that each t-stability on $\mathcal{T}_g$ is induced by some stability data on the abelian category $\mathcal{C}oh\,C_g$. Then we focus on the case $g = 1$ and prove that all the finest t-stabilities on $\mathcal{T}_1$ are exhausted by the standard ones (which depend on a choice of some ordering on a countable union of disjunct copies of the initial curve C_1). As an application of this result, we list down all bounded t-structures on $\mathcal{T}_1$.

It was explained in Proposition 3.3, how to construct a t-stability on the derived category of an abelian category $\mathcal{A}$ by extending some stability data on $\mathcal{A}$ which, in its turn, can be defined by a positive system of additive functions on $K_0(\mathcal{A})$ (see Definition 2.3). In the case $\mathcal{A} = \mathcal{C}oh\,C_g$, there is the standard exhaustive positive system $(\operatorname{rk}, \deg)$, where rk is the rank and deg is the degree of a coherent sheaf. This system provides each coherent sheaf $F \in \mathcal{C}oh\,C_g$ with a slope

$$\gamma(F) = \begin{cases} \dfrac{1}{\pi}\operatorname{arcctg}\left(-\dfrac{\deg F}{\operatorname{rk} F}\right), & \text{if} \quad \operatorname{rk} F \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Since arcctg is strictly decreasing, this slope is equivalent to a slope

$$\bar{\mu}(F) = \begin{cases} \dfrac{\deg F}{\operatorname{rk} F}, & \text{if} \quad \operatorname{rk} F \neq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

which takes values in a (naturally ordered) set

$$\widehat{\mathbb{Q}} = \mathbb{Q} \sqcup \{+\infty\}.$$

A coherent sheaf E is $\bar{\mu}$-semistable, if $\bar{\mu}(F) \leqslant \bar{\mu}(E)$ for any non zero subsheaf $F \subset E$. In particular, all the torsion sheaves are $\bar{\mu}$-semistable and a torsion free sheaf is $\bar{\mu}$-semistable iff it is semistable in the Mumford–Takemoto sense. So, on $\mathcal{C}oh\,C_g$ we have the stability data $\left(\widehat{\mathbb{Q}}, \{\Pi_q\}_{q\in\widehat{\mathbb{Q}}}\right)$ with the semistable subcategories Π_q consisting of all $\bar{\mu}$-semistable sheaves of slope q. This data canonically induces a t-stability on the derived category $\mathcal{T}_g$. For briefness, we call it MT-*stability*. But clearly this is not the finest t-stability.

Proposition 7.1. *If* $g > 0$*, then any* t*-stability* Φ *on* $D^b(\mathcal{C}oh\,C_g)$ *can be refined to a* t*-stability induced by some stability data on* $\mathcal{C}oh\,C_g$*.*

Proof. Since $\mathcal{C}oh\,C_g$ has homological dimension 1, we only need to show that for any sheaf $E \in \mathcal{C}oh\,C_g$ all its Harder–Narasimhan factors $(E_{\varphi_0}, \dots, E_{\varphi_n})$ w. r. t. Φ belong to the abelian core $\mathcal{C}oh\,C_g$. This follows at once from the following

Lemma 7.2. *Let a sheaf $E \in \mathcal{C}oh\, C_g$ be fitted in an exact triangle of $D^b(\mathcal{C}oh\, C_g)$*

$$\begin{tikzcd} & X \arrow[dr, dashed] & \\ E \arrow[ur, "f"] & & Y, \arrow[ll] \end{tikzcd} \tag{7.1}$$

such that $\mathrm{Hom}^{\leqslant 0}(Y,X) = 0$. *Then* $X, Y \in \mathcal{C}oh\, C_g$ *and* $\mathrm{Hom}_{\mathcal{C}oh\, C_g}(Y,X) = 0$.

Proof. Since the homological dimension of $\mathcal{T}_g$ equals 1, we can assume that $X = X_0[0] \oplus X_1[1]$, where X_0, $X_1 \in \mathcal{C}oh\, C_g$ and $f = f_0 \oplus f_1$ with $f_0 \in \mathrm{Hom}_{\mathcal{C}oh\, C_g}(E, X_0)$, $f_1 \in \mathrm{Ext}^1_{\mathcal{C}oh\, C_g}(E, X_1)$. Moreover, $f_i = 0 \Leftrightarrow X_i = 0$ for each i. Let

$$0 \to X_1 \to G \xrightarrow{d_{-1}} E \to 0, \tag{7.2}$$

be an extension defined by f_1 and $G \xrightarrow{d_{-1}} E$ be a complex representing the object $X_1[1]$. Then f_1 is represented by a morphism of complexes:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E \arrow[d, "\mathrm{Id}_E"] \\ G \arrow[r, "d_{-1}"] & E \end{tikzcd}$$

and the direct sum $f_0 \oplus f_{-1}$ is represented by the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E \arrow[d, "\mathrm{Id}_E \oplus f_0"] \\ G \arrow[r, "d_{-1}"] & E \oplus X_0\,. \end{tikzcd}$$

The cone $C(f_0 \oplus f_1)$ can be represented by a complex $E \oplus G \xrightarrow{\delta_{-1}} E \oplus X_0$, whose differential δ_{-1} has the matrix $\begin{pmatrix} \mathrm{Id}_E & d_{-1} \\ f_0 & 0 \end{pmatrix}$. It follows from the commutative diagram:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & X_1 \arrow[r] \arrow[d] & G \arrow[r, "d_{-1}"] \arrow[d] & E \arrow[r] \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & H^{-1} \arrow[r] \arrow[d] & E \oplus G \arrow[r, "\delta_{-1}"] \arrow[d] & E \oplus X_0 \arrow[r] \arrow[d] & H^0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Ker}\, f_0 \arrow[r] \arrow[d] & E \arrow[r, "f_0"] \arrow[d] & X_0 \arrow[r] \arrow[d] & \mathrm{Coker}\, f_0 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & 0 & \end{tikzcd} \tag{7.3}$$

that the cone $C(f_0 \oplus f_1)$ splits as $H^{-1}[1] \oplus \mathrm{Coker}\, f_0[0]$. Therefore, the vanishing condition $\mathrm{Hom}^{\leqslant 0}(Y, X) = 0$ can be rewritten as

$$\mathrm{Hom}^{\leqslant 0}\big(H^{-1}[0] \oplus \mathrm{Coker}\, f_0[-1],\ X_0[0] \oplus X_1[1]\big) = 0.$$

In particular

$$\mathrm{Hom}^0(\mathrm{Coker}\, f_0[-1],\ X_0[0]) = \mathrm{Ext}^1_{\mathcal{C}oh\, C_g}(\mathrm{Coker}\, f_0, X_0) = 0, \tag{7.4}$$

$$\mathrm{Hom}^0(H^{-1}[0],\ X_1[1]) = \mathrm{Ext}^1_{\mathcal{C}oh\, C_g}(H^{-1}, X_1) = 0. \tag{7.5}$$

By the definition of the cokernel, $\mathrm{Hom}_{\mathcal{C}oh\, C_g}(X_0, \mathrm{Coker}\, f_0) = 0$ iff $\mathrm{Coker}\, f_0 = 0$. Since the canonical divisor K_g on the curve of positive genus is either trivial ($g = 0$) or effective ($g > 1$), the space $\mathrm{Hom}_{\mathcal{C}oh\, C_g}(X_0, \mathrm{Coker}\, f_0(K_g))$ is non trivial as soon $\mathrm{Coker}\, f_0) \neq 0$. By the Serre duality, this would mean that $\mathrm{Ext}^1_{\mathcal{C}oh\, C_g}(\mathrm{Coker}\, f_0, X_0) \neq 0$ and contradicts with (7.4). So, $\mathrm{Coker}\, f_0 = 0$.

If $X_1 \neq 0$, then $\mathrm{Hom}_{\mathcal{C}oh\, C_g}(X_1, H^{-1}) \neq 0$ (see diagram (7.3)). Hence, $\mathrm{Hom}_{\mathcal{C}oh\, C_g}(X_1, H^{-1}(K_g)) \neq 0$ and $\mathrm{Ext}^1_{\mathcal{C}oh\, C_g}(H^{-1}, X_1) \neq 0$ by the Serre duality. This contradicts with (7.5).

Thus, $X_1 = 0$ and the triangle (7.1) looks like

$$\begin{tikzcd} & X_0 \arrow[dr, dashed] & \\ E \arrow[ur, "f_0"] & & \mathrm{Ker}\, f_0 . \arrow[ll] \end{tikzcd}$$

This proves the both: Lemma 7.2 and Proposition 7.1.

Unfortunately, a complete description of all the finest t-stabilities on $\mathcal{T}_g$ for $g > 1$ is still unknown to the authors. For elliptic curve such description is given in the next section.

7.1. Description of t-stabilities on $\mathcal{T}_1$. For any rational number q there is Mumford–Takemoto stable vector bundle E with $\bar{\mu}(F) = q$ on an elliptic curve C_1. We write $\mathfrak{M}_q$ for the moduli space of such the bundles. It is well known (see [1]) that $\mathfrak{M}_q$ is isomorphic to the original curve C_1. Moreover,

$$\mathrm{Hom}(F, G) = \mathrm{Hom}(G, F) = \mathrm{Ext}^1(F, G) = \mathrm{Ext}^1(G, F) = 0.$$

for any two distinct bundles $F, G \in \mathfrak{M}_q$, $F \neq G$.

Let us write $\mathfrak{M}_\infty$ for the curve C_1 considered as the moduli space of the skyscrapers $\mathcal{O}_x$. Then we get a set $\mathcal{M}_1$ of Mumford–Takemoto semistable subcategories $\Pi_{(q,F)} = \langle F \rangle$, which depend on $q \in \widehat{\mathbb{Q}}$ and $F \in \mathfrak{M}_q$. If we fix some arbitrary linear order on each $\mathfrak{M}_q$ and extend them lexicographically onto $\mathcal{M}_1$, then we get the finest stability data $(\mathcal{M}_1, \{\Pi_\mu\}_{\mu \in \mathcal{M}_1})$ on the abelian category $\mathcal{C}oh\, C_1$ (the existence of Harder–Narasimhan filtrations is completely evident here).

Further, take the lexicographically ordered set $D(\mathcal{M}_1) = \mathbb{Z} \times \mathcal{M}_1$ as the set of slopes and define the semistable subcategories as $\Pi_{(i,q,F)} = \langle F[i] \rangle$, where $F \in \mathfrak{M}_q$. Then, by Propositions 3.3 and 5.5, each data set $\big(D(\mathcal{M}_1), \{\Pi_\mu\}_{\mu \in D(\mathcal{M}_1)}\big)$ (which depends on the above choices of orders) presents the finest t-stability on $D^b(\mathcal{C}oh\, C_1)$. We call these finest t-stabilities *standard*.

Theorem 7.3. *Any* t*-stability* $(\Phi, \{\Theta_\varphi\}_{\varphi\in\Phi})$ *on* $\mathcal{T}_1$ *can be refined to some standard finest* t*-stability* $\big(D(\mathcal{M}_1), \{\Pi_\mu\}_{\mu\in D(\mathcal{M}_1)}\big)$*. In particular all the finest* t*-stabilities on* $\mathcal{T}_1$ *are standard.*

Proof. Since any t-stability on $\mathcal{T}_1$ is induced by some stability data on the abelian category $\mathcal{C}oh\, C_1$, it is enough to show that each indecomposable coherent sheaf E (which generates the standard semistable subcategory) is Φ-semistable. Suppose the contrary and let E be an Φ-unstable indecomposable coherent sheaf. Then, by Lemma 7.2, E is fitted into exact triple $0 \to Y \to E \to X \to 0$ with $\mathrm{Hom}_{\mathcal{C}oh\, C_g}(Y, X) = 0$. It splits, because of $\mathrm{Ext}^1_{\mathcal{C}oh\, C_g}(X, Y) = 0$ by the Serre duality; thus, $E = X \oplus Y$.

7.2. Description of bounded t-structures on $\mathcal{T}_1$. Any bounded t-structure on $\mathcal{T}_1$ is obtained by some decomposition of the total slope set $\mathcal{M}_1$ and corresponds to certain cotilting torsion pair in $\mathcal{C}oh\, C_1$. We will list all these cotilting pairs modulo tha action of the autoequivalences group $\mathrm{Aut}\, \mathcal{T}_1$. Let $I = \big(\mathbb{Q}\cap[0,1)\big)\cup\{\infty\}$. Then any $q \in I$ and any subset $P \subset \mathfrak{M}_q \simeq C_1$ (the case $P = \varnothing$ is admissible as well) define a cotilting torsion pair $\big(\mathcal{A}(q,P)_1,\, \mathcal{A}(q,P)_0\big)$, where

$$\begin{aligned}
\mathcal{A}(q,P)_0 &= \langle X \mid \bar\mu(X) < q \quad \text{or} \quad \bar\mu(X) = q \quad \text{and} \quad X \in P\rangle,\\
\mathcal{A}(q,P)_1 &= \langle X \mid \bar\mu(X) > q \quad \text{or} \quad \bar\mu(X) = q \quad \text{and} \quad X \notin P\rangle.
\end{aligned}$$

and X is running trough the simple vector bundles on C_1. In particular, the tautological t-structuree corresponds to $q = 0$ and $P = \varnothing$. So, we have got

Theorem 7.4. *All bounded* t*-structures on the bounded derived category of coherent sheaves on a smooth elliptic curve are exhausted (modulo the action of the autoequivalences group* $\mathrm{Aut}\, D^b(\mathcal{C}oh\, C_1)$*) by the* t*-structures coming from cotilting torsion pairs listed above.*

References

1. Atiyah M. F., *Vector Bundles Over an Elliptic Curve*, Proc. Lond. Math. Soc. **VII** (1957), 414–452.
2. Beilinson A. A., *Coherent Sheaves on $\mathbb{P}_n$ and Problems in Linear Algebra*, Funk. An. **12** (1978), 68–69.
3. Bondal A. I., *Helices, Representations of Quivers and Koszul Algebras*, Helices and Vector Bundles: Seminare Rudakov. London Math. Soc., Lect. N. Ser. **148** (1990), 75–95.
4. Bondal A., Orlov D., *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Math. **125** (2001), no. 3, 327–344.
5. Bondal A., Van den Bergh M., *Generators and representability of functors in commutative and noncommutative geometry*, Moscow Math. J. **3** (2003), no. 1; *math.AG/0204218.*
6. Bridgeland T., *Stability conditions on triangulated categories*, math.AG/0212237.
7. Bridgeland T., *Stability conditions on K3-surfaces*, math.AG/0307164.
8. Gelfand S. I., Manin Tu. I., *Methods of homological algebra, I*, ???, ???., 1988.
9. Douglas M. R., *Dirichlet branes, homological mirror symmetry, and stability*, math.AG/0207021.
10. Gieseker D., *On the moduli of vector bundles on an algebraic surface*, Ann. of Math. **106** (1977), 45.
11. Gorodentsev A., Kuleshov S., *Helix theory. Preprint/MPI 2001 (97).*
12. Happel D., Reiten I., Smalo S., *Tilting in abelian categories and quasitilted algebra*, Memoirs of the AMS. V. 575, Amer. Math. Soc., N.Y., 1996.
13. Maruyama M., *Moduli of stable sheaves. I*, J. Math. Kyoto. **17** (1977), 91.
14. Maruyama M., *Moduli of stable sheaves. II*, J. Math. Kyoto. **18** (1978), 557.

15. Rudakov A., *Stability for an abelian category*, J. Algebra. **197** (1997), no. 1, 231–245.

Translated by THE AUTHORS

E-mail address: `gorod@itep.ru`
`kuleshov@mail.miem.edu.ru`
`rudakov@math.ntnu.no`